MISRIMAL NAVAJEE MUNOTH JAIN ENGINEERING COLLEGE, CHENNAI - 97

DEPARTMENT OF MATHEMATICS

MATHEMATICS (MA2111)

FOR

FIRST SEMESTER ENGINEERING STUDENTS
ANNA UNIVERSITY SYLLABUS

This text contains some of the most important short answer (Part A) and long answer (Part B) questions and their answers. Each unit contains 30 university questions. Thus, a total of 150 questions and their solutions are given. A student who studies these model problems will be able to get pass mark (hopefully!!).

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UNIT I MATRICES

SHORT ANSWER

Problem 1. Two eigen values of $A=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6.

Find the eigen values of A^{-1}.

Solution: Sum of the eigen values = Sum of the main diagonal elements = 3 + 5 + 3 = 11.
If λ is the third eigen value, then $3 + 6 + \lambda = 11$. Therefore $\lambda = 2$.
Hence eigen values of A are 2, 3, 6.
The eigen values of A^{-1} are 1/2, 1/3, 1/6

Problem 2. Find the eigen values of A^3, given, $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution: A is an upper triangular matrix.
Hence the eigen values of A are the diagonal elements 1, 2, 3.
The eigen values of A^3 are 1^3, 2^3, 3^3. i.e., 1, 8, 27.

Problem 3. If a, b are the eigen values of $A=\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$, form the matrix whose eigen values are a^3, b^3.

Solution: a^3, b^3 are the eigen values of the matrix A^3.

$$\text{Now } A^2 = A.A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 38 & -50 \\ -50 & 138 \end{bmatrix}$$

Problem 4. Find the sum and product of the eigen values of the matrix

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution: Sum of the eigen values = sum of the main diagonal elements
= -2 + 1 + 0 = -1.

Product of the eigen values $= |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(0-12) - 2(0-6) - 3(-4+1)$

$= 24 + 12 + 9 = 45.$

Problem 5. If the sum of two eigen values and trace of a 3 X 3matrix A are equal, find the value of $|A|$.

Solution: Sum of the eigen values = $\lambda_1 + \lambda_2 + \lambda_3$ = sum of the diagonal elements
= trace of A.

Given $\lambda_1 + \lambda_2$ = trace of A.
i.e., $\lambda_1 + \lambda_2 = \lambda_1 + \lambda_2 + \lambda_3$
Therefore $\lambda_3 = 0$.
Then $|A|$ = Product of the eigen values of A = $\lambda_1\lambda_2\lambda_3 = 0$

Problem 6. Two eigen values of a singular matrix A of order three are 2 and 3. Find the third eigen value.

Solution: Since A is singular matrix, $|A| = 0$.

Product of the eigen values = $|A| = 0$. Two eigen values are 2 and 3. Therefore the third eigen value has to be 0.

Problem 7. State Cayley-Hamilton Theorem

Solution: Every square matrix satisfies its own characteristic equation.

Problem 8. Verify Cayley- Hamilton Theorem for the matrix $A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & -1 \\ -1 & 5-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(5-\lambda) - 1 = 0$$

$$\lambda^2 - 8\lambda + 14 = 0$$

To prove that A satisfies the characteristic equation i.e., $A^2 - 8A + 14I = 0$

$A^2 = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix}$

$A^2 - 8A + 14I = \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix} - 8\begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} + 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 10 & -8 \\ -8 & 26 \end{bmatrix} + \begin{bmatrix} -24 & 8 \\ 8 & -40 \end{bmatrix} + \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 9. If A = $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, write A^2 in terms of A and I, using Cayley- Hamilton Theorem.

Solution: The characteristic equation of A is $| A - \lambda I |= 0$

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & 5-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(5-\lambda) = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

By Cayley-Hamilton Theorem, $A^2 - 6A + 5I = 0$

Therefore, $A^2 = 6A - 5I$

Problem 10. Given A = $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, find A^4 using Cayley-Hamilton Theorem.

Solution: The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$ where

S_1 = sum of the main diagonal elements = 1 + (-1) = 0

$S_2 = |A|$ = 1(-1) -2(2) = -5

The characteristic equation is $\lambda^2 - 5 = 0$

By Cayley-Hamilton Theorem, A satisfies its characteristic equation.

Therefore $A^2 - 5I = 0$

$$A^2 = 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^4 = A^2. A^2 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

Problem 11. If the sum of the eigen values of the matrix of the quadratic form equal to zero, then what will be the nature of the quadratic form?

Solution: Given $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

Case (i) $\lambda_1, \lambda_2, \lambda_3$ cannot be all positive

Case (ii) $\lambda_1, \lambda_2, \lambda_3$ cannot be all negative

Case (iii) Some are positive and some are negative is possible.

Therefore the quadratic form is indefinite.

Problem 12. Show that the matrix P = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution: P = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $P^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$PP^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly, $P^TP = I$. Therefore the given matrix is orthogonal.

Problem 13. Determine the nature of the quadratic form $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

Solution: Matrix of the quadratic form is $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

$D_1 = |1| = 1$; $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$;

$D_3 = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{vmatrix} = 1(18-1) - 1(6-2) + 2(1-6) = 3$

D_1, D_2, D_3 are all positive. Therefore the Q.F is positive definite.

Problem 14. Determine the nature of the quadratic form

$2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_3x_1$

Solution: Matrix of the quadratic form is $A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$

$D_1 = |2| = 2$; $D_2 = \begin{vmatrix} 2 & 6 \\ 6 & 1 \end{vmatrix} = -34$;

$D_3 = |A| = 2(-3-16) - 6(-18-8) - 2(-24+2) = 162$

D_1, D_3 positive and D_2 negative. Therefore the Q.F is indefinite.

Problem 15. Find the rank, index, signature and nature of the Quadratic Form

$0y_1^2 + 3y_2^2 + 14y_3^2$

Solution: The given quadratic form is in the canonical form (C.F).

Rank of the Q.F = No. of terms in the C.F = 2
Index of the Q.F = No. of positive terms in the C.F = 2
Signature of Q.F. = (No. of positive terms) – (No. of negative terms) = 2 - 0 = 2
Nature of the Q.F. is positive semi definite.

LONG ANSWER

Problem 16. Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Solution:

The characteristic equation is $| A - \lambda I | = 0$.

i.e., $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

i.e., $(-2 - \lambda) [-\lambda(1 - \lambda) -12] - 2[-2\lambda - 6] -3[-4 + 1 - \lambda] = 0$

i.e., $(-2 - \lambda) [\lambda^2 - \lambda -12] + 4\lambda + 12 + 9 + 3\lambda = 0$

i.e., $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ (1)

Now, $(-3)^3 + (-3)^2 - 21(-3) - 45 = -27 + 9 + 63 - 45 = 0$

$\therefore$ -3 is a root of equation (1).

Dividing $\lambda^3 + \lambda^2 - 21\lambda - 45$ by $\lambda + 3$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

Remaining roots are given by

$\lambda^2 - 2\lambda - 15 = 0$

i.e., $(\lambda + 3) (\lambda - 5) = 0$

i.e., $\lambda = -3, 5$.

$\therefore$ The eigen values are -3, -3, 5

The eigen vectors of A are given by $$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1 $\lambda = -3$

Now $$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x_1 + 2x_2 - 3x_3 = 0$

Put $x_2 = k_1, x_3 = k_2$

Then $x_1 = 3k_2 - 2k_1$

$\therefore$ The general eigen vectors corresponding to $\lambda = -3$ is $\begin{bmatrix} 3k_2 - 2k_1 \\ k_1 \\ k_2 \end{bmatrix}$

When $k_1 = 0, k_2 = 1$, we get the eigen vector $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

When $k_1 = 1, k_2 = 0$, we get the eigen vector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Hence the two eigen vectors corresponding to $\lambda = -3$ are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

These two eigen vectors corresponding to $\lambda = -3$ are linearly independent.

Case 2 $\lambda = 5$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $-x_1 - 2x_2 - 5x_3 = 0$

$-8x_2 - 16x_3 = 0$

A solution is $x_3 = 1, x_2 = -2, x_1 = -1$

$\therefore$ Eigen vector corresponding to $\lambda = 5$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

Problem 17. Find the characteristic equation of $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$ and verify Cayley-Hamilton Theorem. Hence find the inverse of the matrix.

Solution: Let $A=\begin{bmatrix}1&-1&2\\-2&1&3\\3&2&-3\end{bmatrix}$ $\therefore$ Characteristic eqn. of A is

$\lambda^3-\lambda^2[1+1-3]+\lambda[-9-9-1]+26=0$

i.e $\lambda^3+\lambda^2-19\lambda+26=0$

By **Cayley-Hamilton theorem** $\therefore A^3+A^2-19A+26I=0$.

<u>Verification:</u>

$$\therefore A^2=A.A=\begin{pmatrix}1&-1&2\\-2&1&3\\3&2&-3\end{pmatrix}\begin{pmatrix}1&-1&2\\-2&1&3\\3&2&-3\end{pmatrix}=\begin{pmatrix}9&2&-7\\5&9&-10\\-10&-7&21\end{pmatrix}$$

$$\therefore A^3=A^2.A=\begin{pmatrix}9&2&-7\\5&9&-10\\-10&-7&21\end{pmatrix}\begin{pmatrix}1&-1&2\\-2&1&3\\3&2&-3\end{pmatrix}=\begin{pmatrix}-16&-21&45\\-43&-16&67\\67&45&-104\end{pmatrix}$$

Substituting in the characteristic equation

$$\begin{pmatrix}-16&-21&45\\-43&-16&67\\67&45&-104\end{pmatrix}+\begin{pmatrix}9&2&-7\\5&9&-10\\-10&-7&21\end{pmatrix}-\begin{pmatrix}19&-19&38\\-38&19&57\\57&38&-57\end{pmatrix}+\begin{pmatrix}26&0&0\\0&26&0\\0&0&26\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

Hence verified.

Now to find the inverse of the matrix A, premultiply the characteristic equation by A^{-1}

$\therefore A^2+A-19I+26A^{-1}=0$

$$\therefore A^{-1}=\frac{1}{26}\left(19I-A-A^2\right)$$

$$=\frac{1}{26}\left[\begin{pmatrix}19&0&0\\0&19&0\\0&0&19\end{pmatrix}-\begin{pmatrix}1&-1&2\\-2&1&3\\3&2&-3\end{pmatrix}-\begin{pmatrix}9&2&-7\\5&9&-10\\-10&-7&21\end{pmatrix}\right]=\frac{1}{26}\begin{pmatrix}9&-5&5\\-3&9&7\\7&5&1\end{pmatrix}$$

Problem 18. Given A = $\begin{bmatrix}1&0&3\\2&1&-1\\1&-1&1\end{bmatrix}$**,** use Cayley-Hamilton Theorem to find the inverse of A and also find A^4

Solution:

The characteristic equation of A is

$$\begin{vmatrix}1-\lambda&0&3\\2&1-\lambda&-1\\1&-1&1-\lambda\end{vmatrix}=0$$

i.e., $(1-\lambda)\,[(1-\lambda)(1-\lambda)-1] + 3[-2-(1-\lambda)] = 0$

i.e., $(1-\lambda)^3 - (1-\lambda) - 6 - 3 + 3\lambda = 0$

i.e., $1 - 3\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda - 9 + 3\lambda = 0$

i.e., $-\lambda^3 + 3\lambda^2 + \lambda - 9 = 0$

i.e., $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

To find A^{-1}, multiplying by A^{-1}, $A^2 - 3A - I + 9A^{-1} = 0$

$\therefore \quad A^{-1} = \frac{1}{9}[-A^2 + 3A + I]$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

To find A^4:

We have $A^3 - 3A^2 - A + 9I = 0$

i.e., $A^3 = 3A^2 + A - 9I$ (1)

Multiplying (1) by A, we get,

$A^4 = 3A^3 + A^2 - 9A$ $\therefore$

$= 3(3A^2 + A - 9I) + A^2 - 9A$ using (1)

$= 10A^2 - 6A - 27I$

$$= 10\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

Problem 19. . If $A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$ express $A^6 - 25A^2 + 122A$ as a single matrix

Solution: To avoid higher powers of A like A^6 we use Cayley Hamilton Theorem.

Characteristic equation is $\lambda^3 - 4\lambda^2 + 5\lambda + 2 = 0$

By Cayley Hamilton Theorem $A^3 - 4A^2 + 5A + 2I = 0$

To find $A^6 - 25A^2 + 122A$ we will express this in terms of smaller powers of A using the characteristics equation. We know that (Divisor) X (Quotient) + Remainder = Dividend

Assuming $A^3-4A^2+5A+2I$ as the divisor we get,

$$\begin{array}{r|l} & A^3+4A^2+11A+22I \\ \hline A^3-4A^2+5A+2I & A^6+0A^5+0A^4-25A^2+122A+0I \\ & A^6-4A^5+5A^4+2A^3 \\ \hline & \quad 4A^5-5A^4-2A^3-25A^2+122A \\ & \quad 4A^5-16A^4+20A^3+8A^2 \\ \hline & \quad 11A^4-22A^3-33A^2+122A \\ & \quad 11A^4-44A^3+55A^2+22A \\ \hline & \qquad 22A^3-88A^2+100A \\ & \qquad 22A^3-88A^2+110A+44I \\ \hline & \qquad\qquad -10A-44I \end{array}$$

$$\therefore A^6-25A^2+122A=\left(A^3-4A^2+5A+2I\right)\left(A^3+4A^2+11A+22I\right)+\left(-10A-44I\right)$$

But $A^3-4A^2+5A+2I=0$

$$A^6-25A^2+122A=0-10A-44I$$

$$=-\left(10A+44I\right)$$

$$=-\left[\begin{pmatrix} 0 & 0 & 20 \\ 20 & 10 & 0 \\ -10 & -10 & 20 \end{pmatrix}+\begin{pmatrix} 44 & 0 & 0 \\ 0 & 44 & 0 \\ 0 & 0 & 44 \end{pmatrix}\right]$$

$$=-\begin{pmatrix} 44 & 0 & 20 \\ 20 & 54 & 0 \\ -10 & -10 & 74 \end{pmatrix}$$

$$=-\begin{pmatrix} -44 & 0 & -20 \\ -20 & -54 & 0 \\ -10 & 10 & -74 \end{pmatrix}$$

Problem 20. If λi are the eigen values of the matrix A, then prove that

i $k\lambda i$ are the eigen values of kA where 'k' is a nonzero scalar.

ii. λ_i^m are the eigen value of A^m and

iii. $\frac{1}{\lambda i}$ are the eigen values of A^{-1}.

Solution: Let λi be the eigen values of matrix A and Xi be the corresponding eigen vectors. Then by defn: $AXi=\lambda iXi......(I)$ (i.e by defn. of eigen vectors)

i. Premultiply (I) with the scalar k. Then

$$k\left(AXi\right)=k\left(\lambda iXi\right)$$

$$i.e.\left(kA\right)X_i=\left(k\lambda i\right)Xi$$

$\therefore k\lambda i$ are the eigen values of kA (comparing with (I) i.e by defn.)

ii. Premultiply (I) with A, then

$A(AXi) = A(\lambda iXi)$

$i.e. A^2 X^i = \lambda i(AXi)$

$= \lambda i(\lambda_i Xi)$ from (I)

$= (\lambda i)^2 Xi$

III^{1y} we can prove that $A^3 Xi = (\lambda_i)^3 Xi$ and so on $A^m Xi = (\lambda i)^m Xi$

$\because \lambda i^m$ are the eigen values of the A^m (comparing with (I) i.e. by defn.)

iii. Premultiply (I) with A^{-1}, then

$A^{-1}(AXi) = A^{-1}(\lambda iXi)$

$i.e. (A^{-1}A) Xi = \lambda i (A^{-1}Xi)$

$i.e.\ IXi = \lambda i(A^{-1}Xi)$

$i.e. A^{-1}Xi = \frac{1}{\lambda i} Xi$

$\therefore \frac{1}{\lambda i}$ are the eigen values of A^{-1} (comparing with (I)).

Problem 21. Find the characteristic vectors of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and verify that they are mutually orthogonal.

Solution: A = $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ Characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Solving: $\lambda = 1, 2, 3$

Consider the matrix equation $(A - \lambda I) X = 0$

Case (i) when $\lambda = 1$;

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} i.e. \begin{matrix} 1x_1 + 0x_2 + 1x_3 = 0 - (1) \\ 0x_1 + 1x_2 + 0x_3 = 0 - (2) \\ 1x_1 + 0x_2 + 1x_3 = 0 - (3) \end{matrix}$$

equation (1) & (3) are identical.

Solving (1) and (2) using the rule of cross multiplication

$$\frac{x_1}{0-1} = \frac{x_2}{0-1} = \frac{x_3}{0-1}\ i.e. \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1} \therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Case (ii) when $\lambda = 2$;

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} i.e. \quad \begin{matrix} 0x_1 + 0x_2 + 1x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = 0 \\ 1x_1 + 0x_2 + 0x_3 = 0 \end{matrix} \qquad \begin{matrix} x_3 = 0 \\ i.e.\ x_2 \text{ is arbitrary } say\ k \\ x_1 = 0 \end{matrix}$$

$$\therefore X_2 = \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} i.e \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Case (ii) when $\lambda = 3$;

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} i.e. \quad \begin{matrix} -x_1 + 0x_2 + 1x_3 = 0 \\ 0x_1 + 1x_2 + 0x_3 = 0 \\ 1x_1 + 0x_2 + 1x_3 = 0 \end{matrix} \qquad \text{Solving (1) and (2)}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1} \therefore X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus the eigen values are 1,2,3 and the correspondent eigen vectors are

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \quad \text{To check orthogonallity,} \quad X_1^T X_2 = 0$$

$X_2^T X_3 = 0$

$X_1^T X_3 = 0$

$\therefore X_1, X_2,\ X_3$

are mutually orthogonal.

Problem 22. Find the latent vectors of $\begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$

Solution: Characteristic equation is $(\lambda + 1)^3 = 0 \therefore \lambda = -1, -1, -1$

When $\lambda = -1$ (repeated 3 times) $\therefore$ we have to find 3 corresponding latent vectors.

$$\begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} i.e. \quad \begin{matrix} 7x_1 + 6x_2 + 5x_3 = 0 \\ 14x_1 - 12x_2 + 10x_3 = 0 \\ 7x_1 + 6x_2 + 5x_3 = 0 \end{matrix} \qquad \text{All three equation are identical}$$

.i.e. we get only one equation, but we have to find three vectors that are linearly independent.

$$\therefore \text{Assume } x_1 = 0 \Rightarrow -6x_2 + 5x_3 = 0\, i.e. -6x_2 = -5x_3 \ i.e. \frac{x_2}{5} = \frac{x_3}{6} \therefore X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$$

Assume $x_2 = 0 \Rightarrow -7x_2 + 5x_3 = 0$ *i.e.* $7x_1 = -5x_3$*i.e..* $\frac{x_1}{-5} = \frac{x_3}{7} \therefore X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}$

And assume $x_2 = 0 \Rightarrow 7x_2 - 6x_3 = 0$ *i.e.* $7x_1 = 6x_2$ 0*i.e..* $\frac{x_1}{6} = \frac{x_2}{7} \therefore X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$

X_1, X_2 and X_3 are linearly independent.

Problem 23. Find the eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

Solution:

The characteristic equation of A is $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} = 0$

$$\begin{aligned}
\text{i.e.,}\quad (1-\lambda)[(2-\lambda)(3-\lambda)-4] - 1[0+4] + 1[0+4(2-\lambda)] &= 0 \\
\text{i.e.,}\quad (1-\lambda)(\lambda^2 - 5\lambda + 6 - 4) - 4 + 8 - 4\lambda &= 0 \\
\text{i.e.,}\quad (1-\lambda)(\lambda^2 - 5\lambda + 2) + 4 - 4\lambda &= 0 \\
\text{i.e.,}\quad (1-\lambda)(\lambda^2 - 5\lambda + 2 + 4) &= 0 \\
\text{i.e.,}\quad (\lambda - 1)(\lambda^2 - 5\lambda + 6) &= 0 \\
\text{i.e.,}\quad (\lambda - 1)(\lambda - 2)(\lambda - 3) &= 0
\end{aligned}$$

$\therefore$ The eigen values of A are $\lambda = 1, 2, 3$.

The eigen vectors are given by $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case 1 $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} -4 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-4x_1 + 4x_2 + 2x_3 = 0$$
$$x_2 + x_3 = 0$$

A solution is, $x_3 = 2$, $x_2 = -2$, $x_1 = -1$

$\therefore$ Eigen vector $X_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$

Case 2 $\lambda = 2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0$$
$$x_3 = 0$$

A solution is, $x_3 = 0$, $x_2 = 1$, $x_1 = 1$

$\therefore$ Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Case 3 $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2x_1 + x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$

A solution is, $x_3 = 1$, $x_2 = 1$, $x_1 = 1$

$\therefore$ Eigen vector $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Problem 24. Diagonalise the matrix $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ using orthogonal transformation.

Solution: Characteristic equation is $\lambda^3 - 10\lambda^2 + 27 - 18 = 0$

Solving we get the eigen value as $\lambda = 1, 3, 6$

When $\lambda = 1, X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$; When $\lambda = 3, X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; When $\lambda = 6, X_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Normalizing each vector, we get $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}$

$\therefore$ Normalized Modal Matrix, $N = \begin{pmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix}$. $N' = N^T = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}$,

Then by the orthogonal transformation,

$$N'AN = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 0 & 2/\sqrt{5} \\ 1/\sqrt{5} & 1 & 0 \end{pmatrix}$$. On simplifying, we get

$N'AN = D(\lambda_1, \lambda_2, \lambda_3)$

$= D(1,3,6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ which is diagonal matrix with eigen values along the diagonal (in order).

Problem 25. Reduce $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ to a diagonal matrix by orthogonal reduction.

Solution: Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \therefore \lambda = 8, 2, 2$

When $\lambda = 8$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e $-2x_1 + 2x_2 + 2x_3 = 0$

$-2x_1 - 5x_2 + 1x_3 = 0$

$2x_1 - 1x_2 + 5x_3 = 0$

Solving any two equations $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} \therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

When $\lambda = 2$ (repeated twice)

$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ i.e $-2x_1 + 2x_2 + 2x_3 = 0$. All the equations are identical.

To get one of the vectors, assume $x_1 = 0 \Rightarrow x_2 - x_3 = 0$ *i.e.* $\frac{x_2}{1} = \frac{x_3}{1} \therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$X_1^T X_2 = 0$. Therefore X_1 and X_2 are orthogonal. Now assume $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to be mutually orthogonal with X_1 and X_2.

$$\left.\begin{array}{l} X_1^T X_3 = 0 \ \ i.e.\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \ i.e. 2a - b + c = 0 \\ and\ X_2^T X_3 = 0 \ \ i.e.\begin{pmatrix} 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \ i.e. 0a - b + c = 0 \end{array}\right\} i.e \frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

After normalizing these 3 mutually orthogonal vectors, we get the normalized Modal Matrix $N = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix}$

Diagonalizing we get

$$D = N^T A N = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

on simplifying we get $D = D(\lambda_1, \lambda_2, \lambda_3)$

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= D(8, \ 2, \ 2)$$

Problem 26. Diagonalise the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Solution:

The characteristic equation of A is $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

i.e., $(\lambda-1)(\lambda^2 - 8\lambda + 16) = 0$

$\therefore$ The eigen values of A are $\lambda = 1, 4, 4$.

The eigen vectors are given by $\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case 1 $\lambda = 1$

Eigen vector $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Case 2 $\lambda = 4$

Eigen vector $X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now assume $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to be mutually orthogonal with X_1 and X_2.

$$\left.\begin{aligned} X_1^T X_3 = 0 \ \ i.e. -a + b + c = 0 \\ and\ X_2^T X_3 = 0 \ \ i.e. -b + c = 0 \end{aligned}\right\} \quad i.e \frac{a}{2} = \frac{b}{1} = \frac{c}{1}$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Hence the modal matrix $M = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

The Normalized Modal Matrix is $N = \begin{pmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$

Diagonalizing, we get

$$D = N^T AN = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \mathrm{D}(1, 4, 4)$$

Problem 27. Reduce the Quadratic From $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ into canonical form by orthogonal reduction. Hence find the nature, rank, index and the signature of the Q.F. Find also a nonzero set of values of X which will make the Q.F. vanish.

Solution: Matrix of the given Q.F. is $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & -5 \end{pmatrix}$, which is a real and symmetric matrix. The characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

Solving, we get $\lambda = 0, \quad 3, \quad 14$

When $\lambda = 0, X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$; When $\lambda = 3, X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; When $\lambda = 14, X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

and X_1, X_2, X_3 are mutually orthogonal since $X_1^T, X_2 = 0, X_2^T X_3 = 0 \; and X_3^T X_1 = 0$

Normalizing these vectors we get the normalized model matrix

$$N = \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{pmatrix}$$

Diagonalising we get $D = N^T AN$

$= D(\lambda_1\lambda_2, \lambda_3)$ in order

$= D(0, \quad 3, \quad 14)$

i.e $$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$ (i.e. the eigen values in order along the principal diagonal).

Now to reduce the Q.F to C.F (.i.e Canonical form)

Consider the orthogonal transformation X = NY where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Then the Q.F. $X^T AX$ becomes $(NY)^T A(NY) = Y^T (N^T AN)Y$

$= Y^T DY$ since $N^T AN = D$

$$= (y_1 y_2 y_3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0{y_1}^2 + 3{y_2}^2 + 14{y_3}^2$$

Thus = $0{y_1}^2 + 3{y_2}^2 + 14{y_3}^2$ is the Canonical form of the given Q.F. And the equations of this transformation are got from X= NY.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = NY = \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\therefore x_1 = \frac{1}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 - \frac{3}{\sqrt{14}} y_3$$

$$x_2 = -\frac{5}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 + \frac{3}{\sqrt{14}} y_3$$

$$x_3 = \frac{4}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 - \frac{3}{\sqrt{14}} y_3$$

To get the non-zero set of values of x which make the Q.F zero we assume values for y_1, y_2 and y_3 such that the C.F. vanishes.

i.e $0y_1^{\,2}+3y_2^{\,2}+14y_3^{\,2}$ will vanish if $y_2=0, y_3=0$ and y_1 is any arbitrary value (for simplicity sake, assume y_1 as the denominator of the coeff. of y_1 in the equations) let $y_1=\sqrt{42}$

$$\therefore x_1=\frac{1}{\sqrt{42}}\left(\sqrt{42}\right)+\frac{1}{\sqrt{3}}(0)-\frac{3}{\sqrt{14}}(0)$$

i.e. $x_1=1+0-0=1$

III^{1y} $x_2=-5+0+0=-5$

and $x_3=4+0-0=4$

Thus the set of values of x $i.e(1,\quad -5,\quad 4)$ will reduce the given Q.F. to zero.

To find the rank, index, signature and nature using canonical form:

C.F. is $0y_1^{\,2}+3y_2^{\,2}+14y_3^{\,2}$

$\therefore$ rank is 2 (no. of terms in C.F)
Index is 2 (no. of positive terms)
Signature of Q.F. = (no. of positive terms) – (no. of negative terms) = 2
Nature of the Q.F. is positive semi definite.

Problem 28. Reduce the Q.F. $2xy+2yz+2zx$ into a form of sum of squares. Find the rank, index and signature of it. Find also the nature of the Q.F.

Solution: Matrix of the Q.F. is $A=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$

Characteristic equation is $\lambda^3-3\lambda-2=0$ solving $\lambda=2,-1,-1$

When $\lambda=2, X_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$

When $\lambda=-1$ (repeated twice) we get identical equations as $x_1+x_2+x_3=0$

$$x_1=0\Rightarrow x_2+x_3=0\ \ i.e.\ x_2=-x_3\ i.e. \frac{x_2}{-1}=\frac{x_3}{1}$$

Assume

$$\therefore X_2\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

which is orthogonal with X_1.

Now to find X_3 orthogonal with both X_1 and X_2 assume $X_3=\begin{pmatrix}a\\b\\c\end{pmatrix}$

$$\left.\begin{array}{l} if\ X_2^T X_3 = 0, \quad a+b+c=0 \\ if\ X_2^T X_3 = 0, \quad 0a-b+c=0 \end{array}\right\}$$

$$i.e.\ \frac{a}{2} = \frac{b}{-1} = \frac{c}{-1}$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad i.e. \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

which is orthogonal with X_1 and X_2.

Normalising these vectors we get $N = \begin{pmatrix} 1/\sqrt{3} & 0/\sqrt{2} & -3/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 2/\sqrt{6} \end{pmatrix}$ and $D = N'AN$

$= D(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.Consider the orthonormal transformation X = NY such that Q.F.is reduced to C.F.

The Q.F. is reduced as

$$\begin{aligned} X^T AX &= (NY)^T A(NY) \\ &= Y^T \left(N^T AN\right) Y \\ &= Y^T DY \\ &= (y_1, y_2, y_3,) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \end{aligned}$$

$\therefore$ The C.F. is $2y_1^2 - y_2^2 - y_3^2$
rank of Q.F.is = no. of terms in C.F=3
index of Q.F. = no. of positive terms in C.F. = 1
signature of Q.F. = (no. of positive terms) – (no. of negative terms)
= 1-2 = -1
Nature of the Q.F. is indefinite.

Problem 29. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 + 4x_1x_3 - 8x_2x_3$ to the canonical form by an orthogonal transformation. Find also the rank, index, signature and the nature of the quadratic form.

Solution:

The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The eigen values of this matrix are 0, 3 and 15 and the corresponding eigen vectors are

$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, which are mutually orthogonal.

The normalized modal matrix is $N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

and $N^TAN = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

Now the orthogonal transformation X = NY will reduce the given quadratic form to the canonical form $0y_1^2 + 3y_2^2 + 15y_3^2$.

Also rank = 2, index = 2, signature = 2. The quadratic form is positive semi definite.

Problem 30. Find the orthogonal transformation which reduces the quadratic form $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$ into the canonical form. Determine the rank, index, signature and the nature of the quadratic form.

Solution:

The matrix of the quadratic form is $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

Expanding $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\lambda = 1$ is a root

Dividing $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ by $\lambda - 1$,

$$\begin{array}{|cccc} 1 & -6 & 9 & -4 \\ 0 & 1 & -5 & 4 \\ \hline 1 & -5 & 4 & |\underline{0} \end{array}$$

The remaining roots are given by $\lambda^2 - 5\lambda + 4 = 0$

$\lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0$

i.e., $\lambda = 1, 4$

$\therefore$ The eigen values of A are $\lambda = 4, 1, 1$

Case 1 $\lambda = 4$

The eigen vectors are given by $\begin{bmatrix} 2-4 & -1 & 1 \\ -1 & 2-4 & -1 \\ 1 & -1 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_1 - x_2 - 2x_3 = 0$

$-3x_2 - 3x_3 = 0$

A solution is $x_3 = 1$, $x_2 = -1$, $x_1 = 1$.

$\therefore$ The corresponding eigen vector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Case 2 $\lambda = 1$

The eigen vectors are given by $\begin{bmatrix} 2-1 & -1 & 1 \\ -1 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_1 - x_2 + x_3 = 0$

Put $x_3 = 0$. We get $x_1 = x_2 = 1$. Let $x_1 = x_2 = 1$

$\therefore$ The eigen vector corresponding to $\lambda = 1$ is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

X_1 and X_2 are orthogonal as $X_1^T X_2 = 1\bullet 0 + (-1)\bullet 1 + 1\bullet 1 = 0$.

To find another vector $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ corresponding to $\lambda = 1$ such that it is orthogonal to both X_1 and X_2 and satisfies $x_1 - x_2 + x_3 = 0$

i.e., $X_1.X_3 = 0$, $X_2.X_3 = 0$ and $a - b + c = 0$

i.e., $1.a - 1.b + 1.c = 0$, $1.a + 1.b + 0.c = 0$ and $a - b + c = 0$.

i.e., $a - b + c = 0$ and $a + b = 0$

i.e., $a = -b$ and $c = 2b$

Put $b = 1$, so that $a = -1$, $c = 2$

$\therefore \qquad X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

The modal matrix is $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Hence the normalized modal matrix is $N = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$

$\therefore$ The required orthogonal transformation is X = NY will reduce the given quadratic form to the canonical form.

C.F= $4y_1^2 + y_2^2 + y_3^2$

Rank of the quadratic form = 3, index = 3, signature = 3. The quadratic form is positive definite.