# MISRIMAL NAVAJEE MUNOTH JAIN ENGINEERING COLLEGE, CHENNAI - 97

**DEPARTMENT OF MATHEMATICS**

# MATHEMATICS (MA2111)

**FOR**

**FIRST SEMESTER ENGINEERING STUDENTS**
**ANNA UNIVERSITY SYLLABUS**

This text contains some of the most important long answer questions (Part B) and their answers. Each unit contains 15 university questions. Thus, a total of 75 questions and their solutions are given. A student who studies these model problems will be able to get pass mark (hopefully!!).

**Prepared by the faculty of Department of Mathematics**

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# UNIT I MATRICES

**Problem 1.** Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

**Solution:**

The characteristic equation is $| A - \lambda I | = 0$.

i.e., $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

i.e., $(-2 - \lambda) [-\lambda(1 - \lambda) -12] - 2[-2\lambda - 6] -3[-4 + 1 - \lambda] = 0$

i.e., $(-2 - \lambda) [\lambda^2 - \lambda -12] + 4\lambda + 12 + 9 + 3\lambda = 0$

i.e., $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ (1)

Now, $(-3)^3 + (-3)^2 - 21(-3) - 45 = -27 + 9 + 63 - 45 = 0$

$\therefore$ -3 is a root of equation (1).

Dividing $\lambda^3 + \lambda^2 - 21\lambda - 45$ by $\lambda + 3$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

Remaining roots are given by

$\lambda^2 - 2\lambda - 15 = 0$

i.e., $(\lambda + 3) (\lambda - 5) = 0$

i.e., $\lambda = -3, 5$.

$\therefore$ The eigen values are -3, -3, 5

The eigen vectors of A are given by $$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**Case 1** $\lambda = -3$

Now $$\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_1 + 2x_2 - 3x_3 = 0$

Put $\quad x_2 = k_1, x_3 = k_2$

Then $\quad x_1 = 3k_2 - 2k_1$

$\therefore$ The general eigen vectors corresponding to $\lambda = -3$ is $\begin{bmatrix} 3k_2 - 2k_1 \\ k_1 \\ k_2 \end{bmatrix}$

When $k_1 = 0, k_2 = 1$, we get the eigen vector $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

When $k_1 = 1, k_2 = 0$, we get the eigen vector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Hence the two eigen vectors corresponding to $\lambda = -3$ are $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

These two eigen vectors corresponding to $\lambda = -3$ are linearly independent.

**Case 2** $\lambda = 5$

$$\begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad -x_1 - 2x_2 - 5x_3 = 0$

$\quad -8x_2 - 16x_3 = 0$

A solution is $x_3 = 1, x_2 = -2, x_1 = -1$

$\therefore$ Eigen vector corresponding to $\lambda = 5$ is $\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

**Problem 2.** Find the characteristic equation of $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$ and verify Cayley-Hamilton Theorem. Hence find the inverse of the matrix.

**Solution:** Let $A=\begin{bmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{bmatrix}$ $\therefore$ Characteristic eqn. of A is

$\lambda^3-\lambda^2\left[1+1-3\right]+\lambda\left[-9-9-1\right]+26=0$

i.e $\lambda^3+\lambda^2-19\lambda+26=0$

By **Cayley-Hamilton theorem** $\therefore A^3+A^2-19A+26I=0$.

**Verification:**

$$\therefore A^2=A.A=\begin{pmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{pmatrix}\begin{pmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{pmatrix}=\begin{pmatrix}9 & 2 & -7\\ 5 & 9 & -10\\ -10 & -7 & 21\end{pmatrix}$$

$$\therefore A^3=A^2.A=\begin{pmatrix}9 & 2 & -7\\ 5 & 9 & -10\\ -10 & -7 & 21\end{pmatrix}\begin{pmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{pmatrix}=\begin{pmatrix}-16 & -21 & 45\\ -43 & -16 & 67\\ 67 & 45 & -104\end{pmatrix}$$

Substituting in the characteristic equation

$$\begin{pmatrix}-16 & -21 & 45\\ -43 & -16 & 67\\ 67 & 45 & -104\end{pmatrix}+\begin{pmatrix}9 & 2 & -7\\ 5 & 9 & -10\\ -10 & -7 & 21\end{pmatrix}-\begin{pmatrix}19 & -19 & 38\\ -38 & 19 & 57\\ 57 & 38 & -57\end{pmatrix}+\begin{pmatrix}26 & 0 & 0\\ 0 & 26 & 0\\ 0 & 0 & 26\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Hence verified.

Now to find the inverse of the matrix A, premultiply the characteristic equation by $A^{-1}$

$\therefore A^2+A-19I+26A^{-1}=0$

$$\therefore A^{-1}=\frac{1}{26}\left(19I-A-A^2\right)$$

$$=\frac{1}{26}\left[\begin{pmatrix}19 & 0 & 0\\ 0 & 19 & 0\\ 0 & 0 & 19\end{pmatrix}-\begin{pmatrix}1 & -1 & 2\\ -2 & 1 & 3\\ 3 & 2 & -3\end{pmatrix}-\begin{pmatrix}9 & 2 & -7\\ 5 & 9 & -10\\ -10 & -7 & 21\end{pmatrix}\right]=\frac{1}{26}\begin{pmatrix}9 & -5 & 5\\ -3 & 9 & 7\\ 7 & 5 & 1\end{pmatrix}$$

Problem 3. Given A = $\begin{bmatrix}1 & 0 & 3\\ 2 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}$, use Cayley-Hamilton Theorem to find the inverse of A and also find $A^4$

**Solution:**

The characteristic equation of A is

$$\begin{vmatrix}1-\lambda & 0 & 3\\ 2 & 1-\lambda & -1\\ 1 & -1 & 1-\lambda\end{vmatrix}=0$$

i.e., $(1-\lambda)\,[(1-\lambda)(1-\lambda)-1]+3[-2-(1-\lambda)]=0$

i.e., $(1 - \lambda)^3 - (1 - \lambda) - 6 - 3 + 3\lambda = 0$

i.e., $1 - 3\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda - 9 + 3\lambda = 0$

i.e., $-\lambda^3 + 3\lambda^2 + \lambda - 9 = 0$

i.e., $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley-Hamilton theorem, $A^3 - 3A^2 - A + 9I = 0$

To find $A^{-1}$, multiplying by $A^{-1}$, $A^2 - 3A - I + 9A^{-1} = 0$

$\therefore \quad A^{-1} = \frac{1}{9}[-A^2 + 3A + I]$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ 0 & 2 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

To find $A^4$:

We have $A^3 - 3A^2 - A + 9I = 0$

i.e., $A^3 = 3A^2 + A - 9I$ (1)

Multiplying (1) by A, we get,

$$A^4 = 3A^3 + A^2 - 9A \quad \therefore$$

$$= 3(3A^2 + A - 9I) + A^2 - 9A \quad \text{using (1)}$$

$$= 10A^2 - 6A - 27I$$

$$= 10\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - 27\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix}$$

**Problem 4.** . If $A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$ express $A^6 - 25A^2 + 122A$ as a single matrix

**Solution:** To avoid higher powers of A like $A^6$ we use Cayley Hamilton Theorem.

Characteristic equation is $\lambda^3 - 4\lambda^2 + 5\lambda + 2 = 0$

By Cayley Hamilton Theorem $A^3 - 4A^2 + 5A + 2I = 0$

To find $A^6 - 25A^2 + 122A$ we will express this in terms of smaller powers of A using the characteristics equation. We know that (Divisor) X (Quotient) + Remainder = Dividend

Assuming $A^3 - 4A^2 + 5A + 2I$ as the divisor we get,

$$
\begin{array}{r|l}
 & A^3+4A^2+11A+22I \\
\hline
A^3-4A^2+5A+2I & A^6+0A^5+0A^4-25A^2+122A+0I \\
 & A^6-4A^5+5A^4+2A^3 \\
\hline
 & \quad 4A^5-5A^4-2A^3-25A^2+122A \\
 & \quad 4A^5-16A^4+20A^3+8A^2 \\
\hline
 & \quad 11A^4-22A^3-33A^2+122A \\
 & \quad 11A^4-44A^3+55A^2+22A \\
\hline
 & \qquad 22A^3-88A^2+100A \\
 & \qquad 22A^3-88A^2+110A+44I \\
\hline
 & \qquad\qquad -10A-44I
\end{array}
$$

$$\therefore A^6-25A^2+122A=\left(A^3-4A^2+5A+2I\right)\left(A^3+4A^2+11A+22I\right)+\left(-10A-44I\right)$$

But $A^3-4A^2+5A+2I=0$

$$A^6-25A^2+122A=0-10A-44I$$

$$=-\left(10A+44I\right)$$

$$=-\left[\begin{pmatrix} 0 & 0 & 20 \\ 20 & 10 & 0 \\ -10 & -10 & 20 \end{pmatrix}+\begin{pmatrix} 44 & 0 & 0 \\ 0 & 44 & 0 \\ 0 & 0 & 44 \end{pmatrix}\right]$$

$$=-\begin{pmatrix} 44 & 0 & 20 \\ 20 & 54 & 0 \\ -10 & -10 & 74 \end{pmatrix}$$

$$=-\begin{pmatrix} -44 & 0 & -20 \\ -20 & -54 & 0 \\ -10 & 10 & -74 \end{pmatrix}$$

**Problem 5.** If $\lambda i$ are the eigen values of the matrix A, then prove that

i $k\lambda i$ are the eigen values of $kA$ where 'k' is a nonzero scalar.

ii. $\lambda_i^m$ are the eigen value of $A^m$ and

iii. $\dfrac{1}{\lambda i}$ are the eigen values of $A^{-1}$.

**Solution:** Let $\lambda i$ be the eigen values of matrix A and $Xi$ be the corresponding eigen vectors. Then by defn: $AXi=\lambda iXi......(I)$ ( i.e by defn. of eigen vectors)

i. Premultiply $(I)$ with the scalar k. Then

$$k\left(AXi\right)=k\left(\lambda iXi\right)$$

$$i.e.\left(kA\right)X_i=\left(k\lambda i\right)Xi$$

$\therefore k\lambda i$ are the eigen values of kA (comparing with $(I)$ i.e by defn.)

ii. Premultiply $(I)$ with A, then

$A(AXi) = A(\lambda iXi)$

$i.e. A^2 X^i = \lambda i(AXi)$

$= \lambda i(\lambda_i Xi)$ from (I)

$= (\lambda i)^2 Xi$

III[1y] we can prove that $A^3 Xi = (\lambda_i)^3 Xi$ and so on $A^m Xi = (\lambda i)^m Xi$

$\because \lambda i^m$ are the eigen values of the $A^m$ (comparing with $(I)$ i.e. by defn.)

iii. Premultiply $(I)$ with $A^{-1}$, then

$A^{-1}(AXi) = A^{-1}(\lambda iXi)$

$i.e. (A^{-1}A) Xi = \lambda i(A^{-1}Xi)$

$i.e.\ IXi = \lambda i(A^{-1}Xi)$

$i.e. A^{-1}Xi = \frac{1}{\lambda i} Xi$

$\therefore \frac{1}{\lambda i}$ are the eigen values of $A^{-1}$ (comparing with $(I)$ ).

**Problem 6.** Find the characteristic vectors of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ and verify that they are mutually orthogonal.

**Solution:** A = $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ Characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Solving: $\lambda = 1, 2, 3$

Consider the matrix equation $(A - \lambda I) X = 0$

Case (i) when $\lambda = 1$;

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} i.e. \quad \begin{matrix} 1x_1 + 0x_2 + 1x_3 = 0 - (1) \\ 0x_1 + 1x_2 + 0x_3 = 0 - (2) \\ 1x_1 + 0x_2 + 1x_3 = 0 - (3) \end{matrix}$$

equation (1) & (3) are identical.

Solving (1) and (2) using the rule of cross multiplication

$$\frac{x_1}{0-1} = \frac{x_2}{0-1} = \frac{x_3}{0-1}\ i.e. \frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{1} \therefore X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Case (ii) when $\lambda = 2$;

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \text{ i.e. } \begin{matrix}0x_1+0x_2+1x_3=0\\ 0x_1+0x_2+0x_3=0\\ 1x_1+0x_2+0x_3=0\end{matrix} \qquad \begin{matrix}x_3=0\\ \text{i.e. } x_2 \text{ is arbitrary say } k\\ x_1=0\end{matrix}$$

$$\therefore X_2=\begin{pmatrix}0\\ k\\ 0\end{pmatrix} \text{i.e} \begin{pmatrix}0\\ 1\\ 0\end{pmatrix}.$$

Case (ii) when $\lambda=3$;

$$\begin{pmatrix}-1 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \text{ i.e. } \begin{matrix}-x_1+0x_2+1x_3=0\\ 0x_1+1x_2+0x_3=0\\ 1x_1+0x_2+1x_3=0\end{matrix} \qquad \text{Solving (1) and (2)}$$

$$\frac{x_1}{1}=\frac{x_2}{0}=\frac{x_3}{1} \therefore X_3=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$$

Thus the eigen values are 1,2,3 and the correspondent eigen vectors are

$$\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}, \begin{pmatrix}0\\ 1\\ 0\end{pmatrix} \text{ and } \begin{pmatrix}1\\ 0\\ 1\end{pmatrix}.$$ To check orthogonallity, $X_1^T X_2=0$

$X_2^T X_3=0$

$X_1^T X_3=0$

$\therefore X_1, X_2,\ X_3$

are mutually orthogonal.

**Problem 7.** Find the latent vectors of $\begin{pmatrix}6 & -6 & 5\\ 14 & -13 & 10\\ 7 & -6 & 4\end{pmatrix}$

**Solution:** Characteristic equation is $(\lambda+1)^3=0 \therefore \lambda=-1,-1,-1$

When $\lambda=-1$ (repeated 3 times) $\therefore$ we have to find 3 corresponding latent vectors.

$$\begin{pmatrix}7 & -6 & 5\\ 14 & -12 & 10\\ 7 & -6 & 5\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \text{ i.e. } \begin{matrix}7x_1+6x_2+5x_3=0\\ 14x_1-12x_2+10x_3=0\\ 7x_1+6x_2+5x_3=0\end{matrix} \qquad \text{All three equation are identical}$$

.i.e. we get only one equation, but we have to find three vectors that are linearly independent.

$$\therefore \text{Assume } x_1=0 \Rightarrow -6x_2+5x_3=0 \text{ i.e.} -6x_2=-5x_3 \text{ i.e.} \frac{x_2}{5}=\frac{x_3}{6} \therefore X_1=\begin{pmatrix}0\\ 5\\ 6\end{pmatrix}$$

Assume $x_2 = 0 \Rightarrow -7x_2 + 5x_3 = 0$ *i.e.* $7x_1 = -5x_3 i.e.. \frac{x_1}{-5} = \frac{x_3}{7} \therefore X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}$

And assume $x_2 = 0 \Rightarrow 7x_2 - 6x_3 = 0$ *i.e.* $7x_1 = 6x_2 \; 0 i.e.. \frac{x_1}{6} = \frac{x_2}{7} \therefore X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$

$X_1$, $X_2$ and $X_3$ are linearly independent.

**Problem 8.** Find the eigen vectors of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

**Solution:**

The characteristic equation of A is $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} = 0$

$$\begin{aligned}
\text{i.e., } (1-\lambda)[(2-\lambda)(3-\lambda)-4] - 1[0+4] + 1[0+4(2-\lambda)] &= 0 \\
\text{i.e., } (1-\lambda)(\lambda^2 - 5\lambda + 6 - 4) - 4 + 8 - 4\lambda &= 0 \\
\text{i.e., } (1-\lambda)(\lambda^2 - 5\lambda + 2) + 4 - 4\lambda &= 0 \\
\text{i.e., } (1-\lambda)(\lambda^2 - 5\lambda + 2 + 4) &= 0 \\
\text{i.e., } (\lambda-1)(\lambda^2 - 5\lambda + 6) &= 0 \\
\text{i.e., } (\lambda-1)(\lambda-2)(\lambda-3) &= 0
\end{aligned}$$

$\therefore$ The eigen values of A are $\lambda = 1, 2, 3$.

The eigen vectors are given by $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**Case 1** $\lambda = 1$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} -4 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-4x_1 + 4x_2 + 2x_3 = 0$$
$$x_2 + x_3 = 0$$

A solution is, $x_3 = 2$, $x_2 = -2$, $x_1 = -1$

$\therefore$ Eigen vector $X_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$

**Case 2** $\lambda = 2$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0$$
$$x_3 = 0$$

A solution is, $x_3 = 0$, $x_2 = 1$, $x_1 = 1$

$\therefore$ Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

**Case 3** $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-2x_1 + x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$

A solution is, $x_3 = 1$, $x_2 = 1$, $x_1 = 1$

$\therefore$ Eigen vector $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

**Problem 9.** Diagonalise the matrix $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ using orthogonal transformation.

**Solution:** Characteristic equation is $\lambda^3 - 10\lambda^2 + 27 - 18 = 0$
Solving we get the eigen value as $\lambda = 1, 3, 6$

When $\lambda = 1, X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$; When $\lambda = 3, X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$; When $\lambda = 6, X_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Normalizing each vector, we get $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{pmatrix}$

$\therefore$ Normalized Modal Matrix, $N = \begin{pmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{pmatrix}$. $N' = N^T = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix}$,

Then by the orthogonal transformation,

$$N'AN = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 0 & 2/\sqrt{5} \\ 1/\sqrt{5} & 1 & 0 \end{pmatrix}$$

. On simplifying, we get

$$N'AN = D(\lambda_1, \lambda_2, \lambda_3)$$

$= D(1,3,6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ which is diagonal matrix with eigen values along the diagonal (in order).

**Problem 10.** Reduce $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ to a diagonal matrix by orthogonal reduction.

**Solution:** Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \therefore \lambda = 8, 2, 2$

When $\lambda = 8$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

i.e
$$-2x_1 + 2x_2 + 2x_3 = 0$$
$$-2x_1 - 5x_2 + 1x_3 = 0$$
$$2x_1 - 1x_2 + 5x_3 = 0$$

Solving any two equations $\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} \therefore X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

When $\lambda = 2$ (repeated twice)

$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ i.e $-2x_1 + 2x_2 + 2x_3 = 0$. All the equations are identical.

To get one of the vectors, assume $x_1 = 0 \Rightarrow x_2 - x_3 = 0$ *i.e.* $\frac{x_2}{1} = \frac{x_3}{1}$ $\therefore X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$X_1^T X_2 = 0$. Therefore $X_1$ and $X_2$ are orthogonal. Now assume $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to be mutually orthogonal with $X_1$ and $X_2$.

$$\left.\begin{array}{l} X_1^T X_3 = 0 \;\; i.e. \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \;\; i.e. 2a - b + c = 0 \\ and \; X_2^T X_3 = 0 \;\; i.e. \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \;\; i.e. 0a - b + c = 0 \end{array}\right\} i.e \frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

$$\therefore X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

After normalizing these 3 mutually orthogonal vectors, we get the normalized Modal

Matrix $N = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{pmatrix}$

Diagonalizing we get

$$D = N^T A N = \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

on simplifying we get $D = D(\lambda_1, \lambda_2, \lambda_3)$

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= D(8, \;\; 2, \;\; 2)$$

**Problem 11.** Diagonalise the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

**Solution:**

The characteristic equation of A is $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$

i.e., $(\lambda-1)(\lambda^2 - 8\lambda + 16) = 0$

$\therefore$ The eigen values of A are $\lambda = 1, 4, 4$.

The eigen vectors are given by $\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

**Case 1** $\lambda = 1$

Eigen vector $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

**Case 2** $\lambda = 4$

Eigen vector $X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now assume $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ to be mutually orthogonal with $X_1$ and $X_2$.

$$\left.\begin{aligned} X_1^T X_3 = 0 \ \ i.e. -a + b + c = 0 \\ and\ X_2^T X_3 = 0 \ \ i.e. -b + c = 0 \end{aligned}\right\} \quad i.e \frac{a}{2} = \frac{b}{1} = \frac{c}{1}$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Hence the modal matrix $M = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

The Normalized Modal Matrix is $N = \begin{pmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$

Diagonalizing, we get

$$D = N^T AN = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \text{D}(1, 4, 4)$$

**Problem 12.** Reduce the Quadratic From $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$ into canonical form by orthogonal reduction. Hence find the nature, rank, index and the signature of the Q.F. Find also a nonzero set of values of X which will make the Q.F. vanish.

**Solution:** Matrix of the given Q.F. is $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & -5 \end{pmatrix}$, which is a real and symmetric matrix. The characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

Solving, we get $\lambda = 0, \quad 3, \quad 14$

When $\lambda = 0, X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$; When $\lambda = 3, X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; When $\lambda = 14, X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

and $X_1, X_2, X_3$ are mutually orthogonal since $X_1^T, X_2 = 0, X_2^T X_3 = 0$ $and X_3^T X_1 = 0$

Normalizing these vectors we get the normalized model matrix

$$N = \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{pmatrix}$$

Diagonalising we get $D = N^T AN$

$= D(\lambda_1 \lambda_2, \lambda_3)$ in order

$= D(0, \ 3, \ 14)$

i.e $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$ (i.e. the eigen values in order along the principal diagonal).

Now to reduce the Q.F to C.F (.i.e Canonical form)

Consider the orthogonal transformation X = NY where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Then the Q.F. $X^T AX$ becomes $(NY)^T A(NY) = Y^T (N^T AN)Y$

$= Y^T DY$ since $N^T AN = D$

$$= (y_1 y_2 y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0{y_1}^2 + 3{y_2}^2 + 14{y_3}^2$$

Thus = $0{y_1}^2 + 3{y_2}^2 + 14{y_3}^2$ is the Canonical form of the given Q.F. And the equations of this transformation are got from X= NY.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = NY = \begin{pmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\therefore x_1 = \frac{1}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 - \frac{3}{\sqrt{14}} y_3$$

$$x_2 = -\frac{5}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 + \frac{3}{\sqrt{14}} y_3$$

$$x_3 = \frac{4}{\sqrt{42}} y_1 + \frac{1}{\sqrt{3}} y_2 - \frac{3}{\sqrt{14}} y_3$$

To get the non-zero set of values of x which make the Q.F zero we assume values for $y_1, y_2$ and $y_3$ such that the C.F. vanishes.

i.e $0y_1^{\,2} + 3y_2^{\,2} + 14y_3^{\,2}$ will vanish if $y_2 = 0, y_3 = 0$ and $y_1$ is any arbitrary value (for simplicity sake, assume $y_1$ as the denominator of the coeff. of $y_1$ in the equations) let $y_1 = \sqrt{42}$

$\therefore x_1 = \frac{1}{\sqrt{42}}\left(\sqrt{42}\right) + \frac{1}{\sqrt{3}}(0) - \frac{3}{\sqrt{14}}(0)$

*i.e.* $x_1 = 1 + 0 - 0 = 1$

$III^{1y}$ $x_2 = -5 + 0 + 0 = -5$

and $x_3 = 4 + 0 - 0 = 4$

Thus the set of values of $x$ $i.e(1, \quad -5, \quad 4)$ will reduce the given Q.F. to zero.

To find the rank, index, signature and nature using canonical form:

C.F. is $0y_1^{\,2} + 3y_2^{\,2} + 14y_3^{\,2}$

$\therefore$ rank is 2 (no. of terms in C.F)
Index is 2 (no. of positive terms)
Signature of Q.F. = ( no. of positive terms) – (no. of negative terms) = 2
Nature of the Q.F. is positive semi definite.

**Problem 13.** Reduce the Q.F. $2xy + 2yz + 2zx$ into a form of sum of squares. Find the rank, index and signature of it. Find also the nature of the Q.F.

**Solution:** Matrix of the Q.F. is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Characteristic equation is $\lambda^3 - 3\lambda - 2 = 0$ solving $\lambda = 2, -1, -1$

When $\lambda = 2, X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

When $\lambda = -1$ (repeated twice) we get identical equations as $x_1 + x_2 + x_3 = 0$

$$x_1 = 0 \Rightarrow x_2 + x_3 = 0 \;\; i.e.\; x_2 = -x_3 \;\; i.e. \frac{x_2}{-1} = \frac{x_3}{1}$$

Assume

$$\therefore X_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

which is orthogonal with $X_1$.

Now to find $X_3$ orthogonal with both $X_1$ and $X_2$ assume $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\left.\begin{array}{l} if\ X_2^T X_3 = 0, \quad a+b+c=0 \\ if\ X_2^T X_3 = 0, \quad 0a-b+c=0 \end{array}\right\}$$

$$i.e.\ \frac{a}{2} = \frac{b}{-1} = \frac{c}{-1}$$

$$\therefore X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad i.e. \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

which is orthogonal with $X_1$ and $X_2$.

Normalising these vectors we get $N = \begin{pmatrix} 1/\sqrt{3} & 0/\sqrt{2} & -3/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 2/\sqrt{6} \end{pmatrix}$ and $D = N'AN$

$= D(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.Consider the orthonormal transformation X = NY

such that Q.F.is reduced to C.F.

The Q.F. is reduced as

$$X^T AX = (NY)^T A(NY)$$
$$= Y^T (N^T AN) Y$$
$$= Y^T DY$$
$$= (y_1, y_2, y_3,) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$\therefore$ The C.F. is $2y_1^2 - y_2^2 - y_3^2$

rank of Q.F.is = no. of terms in C.F=3

index of Q.F. = no. of positive terms in C.F. = 1

signature of Q.F. = ( no. of positive terms) – (no. of negative terms)

= 1-2 = -1

Nature of the Q.F. is indefinite.

Problem 14. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 + 4x_1x_3 - 8x_2x_3$ to the canonical form by an orthogonal transformation. Find also the rank, index, signature and the nature of the quadratic form.

**Solution:**

The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The eigen values of this matrix are 0, 3 and 15 and the corresponding eigen vectors are

$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, which are mutually orthogonal.

The normalized modal matrix is $N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$

and $N^T AN = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

Now the orthogonal transformation X = NY will reduce the given quadratic form to the canonical form $0y_1^2 + 3y_2^2 + 15y_3^2$.

Also rank = 2, index = 2, signature = 2. The quadratic form is positive semi definite.

**Problem 15.** Find the orthogonal transformation which reduces the quadratic form $2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$ into the canonical form. Determine the rank, index, signature and the nature of the quadratic form.

**Solution:**

The matrix of the quadratic form is $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation of A is $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

Expanding $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

$\lambda = 1$ is a root

Dividing $\lambda^3 - 6\lambda^2 + 9\lambda - 4$ by $\lambda - 1$,

$$\begin{array}{|cccc} 1 & -6 & 9 & -4 \\ 0 & 1 & -5 & 4 \\ \hline 1 & -5 & 4 & \underline{|0} \end{array}$$

The remaining roots are given by $\lambda^2 - 5\lambda + 4 = 0$

$\lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0$

i.e., $\lambda = 1, 4$

$\therefore$ The eigen values of A are $\lambda$ = 4, 1, 1

**Case 1** $\lambda = 4$

The eigen vectors are given by $\begin{bmatrix} 2-4 & -1 & 1 \\ -1 & 2-4 & -1 \\ 1 & -1 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x_1 - x_2 - 2x_3 = 0$

$-3x_2 - 3x_3 = 0$

A solution is $x_3 = 1$, $x_2 = -1$, $x_1 = 1$.

$\therefore$ The corresponding eigen vector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

**Case 2** $\lambda = 1$

The eigen vectors are given by $\begin{bmatrix} 2-1 & -1 & 1 \\ -1 & 2-1 & -1 \\ 1 & -1 & 2-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $x_1 - x_2 + x_3 = 0$

Put $x_3 = 0$. We get $x_1 = x_2 = 1$. Let $x_1 = x_2 = 1$

$\therefore$ The eigen vector corresponding to $\lambda = 1$ is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$X_1$ and $X_2$ are orthogonal as $X_1^T X_2 = 1\bullet 0 + (-1)\bullet 1 + 1\bullet 1 = 0$.

To find another vector $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ corresponding to $\lambda = 1$ such that it is orthogonal to both

$X_1$ and $X_2$ and satisfies $x_1 - x_2 + x_3 = 0$

i.e., $X_1.X_3 = 0$, $X_2.X_3 = 0$ and $a - b + c = 0$

i.e., $1.a - 1.b + 1.c = 0$, $1.a + 1.b + 0.c = 0$ and $a - b + c = 0$.

i.e., $a - b + c = 0$ and $a + b = 0$

i.e., $a = -b$ and $c = 2b$

Put $b = 1$, so that $a = -1$, $c = 2$

$\therefore \qquad X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

The modal matrix is $\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Hence the normalized modal matrix is $N = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$

$\therefore$ The required orthogonal transformation is X = NY will reduce the given quadratic form to the canonical form.

C.F= $4y_1^2 + y_2^2 + y_3^2$

Rank of the quadratic form = 3, index = 3, signature = 3. The quadratic form is positive definite.