MISRIMAL NAVAJEE MUNOTH JAIN ENGINEERING COLLEGE, CHENNAI
DEPARTMENT OF MATHEMATICS
PROBABILITY AND RANDOM PROCESSES (MA2261)
SEMESTER –IV
UNIT-V: LINEAR SYSTEM RANDOM INPUTS
QUESTION BANK ANSWERS

PART-A

Problem 1. If the system function of a convolution type of linear system is given by $h(t)=\begin{cases}\frac{1}{2a} & for\ |t|\le a\\ 0 & for\ |t|>a\end{cases}$ find the relation between power spectrum density function of the input and output processes.

Solution:

$$H(\omega)=\int_{-a}^{a}h(t)e^{-i\omega t}dt=\frac{\sin a\omega}{a\omega}$$

We know that $S_{YY}(\omega)=|H(\omega)|^2 S_{XX}(\omega)$

$$\Rightarrow S_{YY}(\omega)=\frac{\sin^2 a\omega}{a^2\omega^2}S_{XX}(\omega).$$

Problem 2. Give an example of cross-spectral density.

Solution:

The cross-spectral density of two processes $X(t)$ and $Y(t)$ is given by $S_{XY}(\omega)=\begin{cases}p+iq\omega, & if\ |\omega|<1\\ 0, & otherwise\end{cases}$

Problem 3. If a random process $X(t)$ is defined as $X(t)=\begin{cases}A, & 0\le t\le 1\\ 0, & otherwise\end{cases}$, where A is a random variable uniformly distributed from $-\theta\ to\ \theta$. Prove that autocorrelation function of $X(t)$ is $\frac{\theta^2}{3}$.

Solution:

$$R_{XX}(t,t+\tau)=E\left[X(t).X(t+\tau)\right]$$
$$=E\left[A^2\right]\quad\left[\because X(t)\ is\ \ cons\tan t\right]$$

But A is uniform in $(-\theta,\theta)$

$$\therefore f(\theta)=\frac{1}{2\theta},-\theta<a<\theta$$

$$\therefore R_{XX}(t,t+\tau)=\int_{-\theta}^{\theta}a^2 f(a)da$$

$$=\int_{-\theta}^{\theta}a^2.\frac{1}{2\theta}d\theta=\frac{1}{2\theta}\left[\frac{a^3}{3}\right]_{-\theta}^{\theta}$$

$$= \frac{1}{6\theta}\left[\theta^3 - (-\theta)^3\right] = \frac{1}{6\theta}.2\theta^3 = \frac{\theta^2}{3}$$

Problem 4. Check whether $\frac{1}{1+9\tau^2}$ is a valid autocorrelation function of a random process.

Solution: Given $R(\tau) = \frac{1}{1+9\tau^2}$

$$\therefore R(-\tau) = \frac{1}{1+9(-\tau^2)} = \frac{1}{1+9\tau^2} = R(\tau)$$

$\therefore R(\tau)$ is an even function. So it can be the autocorrelation function of a random process.

Problem 5. Find the mean square value of the process X(t) whose power density spectrum is $\frac{4}{4+\omega^2}$.

Solution:

Given $S_{XX}(\omega) = \frac{4}{4+\omega^2}$

Then $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\tau\omega}d\omega$

Mean square value of the process is $E\left[X^2(t)\right] = R_{XX}(0)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{4+\omega^2}d\omega$$

$$= \frac{4}{\pi}\int_{0}^{\infty} \frac{1}{4+\omega^2}d\omega \qquad \left[\because \frac{1}{4+\omega^2} \text{ is even}\right]$$

$$= \frac{4}{\pi}.\frac{1}{2}\left[\tan^{-1}\frac{\omega}{2}\right]_0^{\infty} = \frac{2}{\pi}\left(\tan^{-1}\infty - \tan^{-1}0\right)$$

$$= \frac{2}{\pi}.\frac{\pi}{2} = 1$$

Problem 6. A Circuit has an impulse response given by $h(t) = \begin{cases} \frac{1}{T}; & 0 \le t \le T \\ 0 & ; elsewhere \end{cases}$ find the relation between the power spectral density functions of the input and output processes.

Solution:

$$H(\omega) = \int_0^T h(t)e^{-i\omega t}dt$$

$$= \int_0^T \frac{1}{T} e^{-i\omega t} dt$$

$$= \frac{1}{T}\left[\frac{e^{-i\omega t}}{-i\omega}\right]_0^T$$

$$= \frac{1}{T}\left[\frac{-e^{-i\omega T}+1}{i\omega}\right]$$

$$= \frac{\left(1-e^{-i\omega T}\right)}{Ti\omega}$$

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$= \frac{\left(1-e^{-i\omega T}\right)^2}{\omega^2 T^2} S_{XX}(\omega)$$

Problem 7. Describe a linear system.

Solution:

Given two stochastic process $\{X_1(t)\}$ and $\{X_2(t)\}$, we say that L is a linear transformation if

$$L\left[a_1 X_1(t) + a_2 X_2(t)\right] = a_1 L\left[X_1(t)\right] + a_2 L\left[X_2(t)\right]$$

Problem 8. Given an example of a linear system.

Solution:

Consider the system f with output $tx(t)$ for an input signal $x(t)$.

i.e . $y(t) = f\left[X(t)\right] = t\,x(t)$

Then the system is linear.

For any two inputs $x_1(t), x_2(t)$ the outputs are $tx_1(t)$ and $tx_2(t)$ Now

$$f\left[a_1 x_1(t) + a_2 x_2(t)\right] = t\left[a_1 x_1(t) + a_2 x_2(t)\right]$$

$$= a_1 t x_1(t) + a_2 t x_2(t)$$

$$= a_1 f\left(x_1(t)\right) + a_2 f\left(x_2(t)\right)$$

$\therefore$ the system is linear.

Problem 9. Define a system, when it is called a linear system?

Solution:

Mathematically, a system is a functional relation between input $x(t)$ and output $y(t)$. Symbolically, $y(t) = f\left[x(t)\right], -\infty < t < \infty$..

The system is said to be linear if for any two inputs $x_1(t)$ and $x_2(t)$ and constants a_1, a_2, $f\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 f\left[x_1(t)\right] + a_2 f\left[x_2(t)\right]$.

Problem 10. State the properties of a linear system.

Solution:

Let $X_1(t)$ and $X_2(t)$ be any two processes and a and b be two constants.

If L is a linear filter then
$L\left[a_1 x_1(t)+a_2 x_2(t)\right]=a_1 L\left[x_1(t)\right]+a_2 L\left[x_2(t)\right]$.

Problem 11. Describe a linear system with an random input.

Solution:

We assume that $X(t)$ represents a sample function of a random process $\{X(t)\}$, the system produces an output or response $Y(t)$ and the ensemble of the output functions forms a random process $\{Y(t)\}$. The process $\{Y(t)\}$ can be considered as the output of the system or transformation f with $\{X(t)\}$ as the input the system is completely specified by the operator f.

Problem 12. State the convolution form of the output of linear time invariant system.

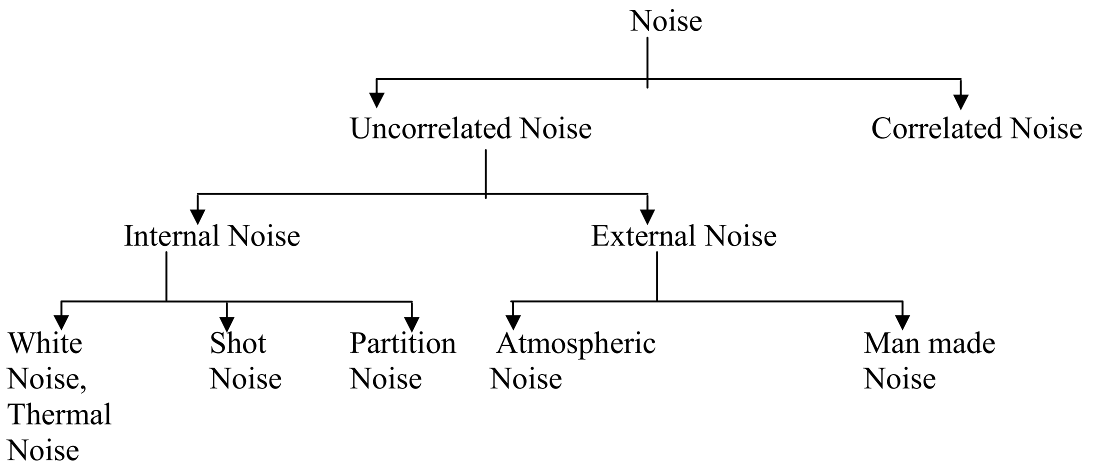
Solution:

If $X(t)$ is the input and $h(t)$ be the system weighting function and $Y(t)$ is the output, then $Y(t)=h(t)*X(t) = \int_{-\infty}^{\infty} h(u)X(t-u)du$

Problem 13. Write a note on noise in communication system.

Solution:

The term noise is used to designate unwanted signals that tend to disturb the transmission and processing of signal in communication systems and over which we have incomplete control.



Problem 14. Define band-limited white noise.

Solution:

Noise with non-zero and constant density over a finite frequency band is called band-limit white noise i.e.,

$$S_{NN}(\omega)=\begin{cases}\dfrac{N_0}{2}, & |\omega|\le \omega_B\\ 0 & , \ otherwise\end{cases}$$

Problem 15. Define (a) Thermal Noise (b) White Noise.

Solution:

(a) Thermal Noise: This noise is due to the random motion of free electrons in a conducting medium such as a resistor.

(or)

Thermal noise is the name given to the electrical noise arising from the random motion of electrons in a conductor.

(b) White Noise(or) Gaussian Noise: The noise analysis of communication systems is based on an idealized form of noise called White Noise.

PART-B

Problem 16. A random process $X(t)$ is the input to a linear system whose impulse response is $h(t)=2e^{-t}, t\geq 0$. If the autocorrelation function of the process is $R_{XX}(\tau)=e^{-2|\tau|}$, find the power spectral density of the output process $Y(t)$.

Solution:

Given $X(t)$ is the input process to the linear system with impulse response $h(t)=2e^{-t}, t\geq 0$

So the transfer function of the linear system is its Fourier transform

$$H(\omega)=\int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$$

$$=\int_{-\infty}^{\infty} 2e^{-t}e^{-i\omega t}dt \qquad \left[\because 2e^{-t}, t\geq 0\right]$$

$$=2\int_{0}^{\infty} e^{-(1+i\omega)t}dt$$

$$=2\left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right]_0^{\infty}$$

$$=\frac{-2}{1+i\omega}[0-1]=\frac{2}{1+i\omega}$$

Given $R_{XX}(\tau)=e^{-2|\tau|}$

$\therefore$ the spectral density of the input is

$$S_{XX}(\omega)=\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}dt$$

$$=\int_{-\infty}^{\infty} e^{-2|\tau|}e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{0} e^{2\tau}e^{-i\omega\tau}d\tau+\int_{0}^{\infty} e^{-2\tau}e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{0} e^{(2-i\omega)\tau}d\tau+\int_{0}^{\infty} e^{-(2+i\omega)\tau}d\tau$$

$$= \left[\frac{e^{(2-i\omega)\tau}}{2-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(2-i\omega)\tau}}{-(2+i\omega)}\right]_{0}^{\infty}$$
$$= \frac{1}{2-i\omega}[1-0] - \frac{1}{2+i\omega}[0-1]$$
$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega}$$
$$= \frac{2+i\omega+2-i\omega}{(2+i\omega)(2-i\omega)} = \frac{4}{4+\omega^2}$$

We know the power spectral density of the output process $Y(t)$ is given by

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$
$$= \left|\frac{2}{1+i\omega}\right|^2 \frac{4}{4+\omega^2}$$
$$= \frac{4}{(1+\omega^2)} \frac{4}{4+\omega^2}$$
$$= \frac{16}{(1+\omega^2)(4+\omega^2)}$$

Problem 17. If $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$, where A is a constant , θ is a random variable with uniform distribution in $(-\pi, \pi)$ and $N(t)$ is a band-limited Gaussian white noise with a power spectral density $S_{NN}(\omega) = \begin{cases} \frac{N_0}{2}, & for \ \ |\omega - \omega_0| < \omega_B \\ 0, & elsewhere \end{cases}$. Find the power spectral density of $Y(t)$. Assume that $N(t)$ and θ are independent.

Solution:

Given $Y(t) = A\cos(\omega_0 t + \theta) + N(t)$

$N(t)$ is a band-limited Gaussian white noise process with power spectral density $S_{NN}(\omega) = \frac{N_0}{2}, |\omega - \omega_0| < \omega_B \ \ ie. \ \omega_0 - \omega_B < \omega < \omega_0 + \omega_B$

Required $S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau) e^{-i\omega\tau} d\tau$

Now $R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$

$$= E\{[A\cos(\omega_0 t + \theta) + N(t)][A\cos(\omega_0 t + \omega_0\tau + \theta) + N(t+\tau)]\}$$

$$= E\left\{\begin{matrix} A^2\cos(\omega_0 t+\theta)\cos(\omega_0 t+\omega_0\tau+\theta)+N(t)N(t+\tau)+A\cos(\omega_0 t+\theta)N(t+\tau) \\ +A\cos(\omega_0 t+\omega_0\tau+\theta)N(t) \end{matrix}\right\}$$

$$= A^2E\left[\cos(\omega_0 t+\theta).\cos(\omega_0 t+\omega_0\tau+\theta)\right]+E\left[N(t)N(t+\tau)\right]$$

$$+AE\left[\cos(\omega_0 t+\theta)\right]E\left[N(t+\tau)\right]$$

$$+AE\left[\cos(\omega_0 t+\omega_0+\theta)\right]E\left[N(t)\right] \quad [\because \theta \text{ and } N(t) \text{ are independent}]$$

$$= \frac{A^2}{2}\left\{E\left[\cos(2\omega_0 t+\omega_0 t+2\theta)\right]+\cos\omega_0\tau\right\}+R_{NN}(\tau)$$

$$+A\,E\left[\cos(\omega_0 t+\theta)\right]E\left[N(t+\tau)\right] \quad +A\,E\left[\cos(\omega_0 t+\omega_0\tau+2\theta)\right]E\left[N(t)\right]\Big\}$$

Since θ is uniformly distributed in $(-\pi,\pi)$ the pdf of θ is $f(\theta)=\frac{1}{2\pi}, -\pi<\theta<\pi$

$$\therefore E\left[\cos(\omega_0 t+\theta)\right]=\int_{-\pi}^{\pi}\cos(\omega_0 t+\theta)f(\theta)\,d\theta$$

$$=\int_{-\pi}^{\pi}\left[\cos\omega_0 t.\cos\theta-\sin\omega_0 t.\sin\theta\right]\frac{1}{2\pi}d\theta$$

$$=\frac{1}{2\pi}\left[\cos\omega_0 t\int_{-\pi}^{\pi}\cos\theta\,d\theta-\sin\omega_0 t\int_{-\pi}^{\pi}\sin\theta\,d\theta\right]$$

$$=\frac{1}{2\pi}\left[\cos\omega_0 t\left[\sin\theta\right]_{-\pi}^{\pi}-\sin\omega_0 t.0\right]=0$$

Similarly $E\left[\cos(2\omega_0 t+\omega_0\tau+2\theta)\right]=\int_{-\pi}^{\pi}\cos(2\omega_0 t+\omega_0\tau+2\theta)\frac{1}{2\pi}d\theta$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(2\omega_0 t+\omega_0\tau)\cos 2\theta-\sin(2\omega_0 t+\omega_0\tau)\sin 2\theta\right]d\theta$$

$$=\frac{1}{2\pi}\left\{\cos(2\omega_0 t+\omega_0\tau)\int_{-\pi}^{\pi}\cos 2\theta\,d\theta-\sin(2\omega_0 t+\omega_0\tau)\int_{-\pi}^{\pi}\sin 2\theta\,d\theta\right\}$$

$$=\frac{1}{2\pi}\left\{\cos(2\omega_0 t+\omega_0\tau).\left[\frac{\sin 2\theta}{2}\right]_{-\pi}^{\pi}-\sin(2\omega_0 t+\omega_0\tau).0\right\}=0$$

$$\therefore R_{YY}(\tau)=\frac{A^2}{2}\cos\omega_0\tau+R_{NN}(\tau)$$

$$\therefore S_{YY}(\omega)=\int_{-\infty}^{\infty}\left[\frac{A^2}{2}\cos\omega_0\tau+R_{NN}(\tau)\right]e^{-i\omega\tau}d\tau$$

$$=\frac{A^2}{2}\int_{-\infty}^{\infty}\cos\omega_0\tau.e^{-i\omega\tau}d\tau+\int_{-\infty}^{\infty}R_{NN}(\tau)e^{-i\omega\tau}d\tau$$

$$=\frac{\pi A^2}{2}\left\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right\}+S_{NN}(\omega)$$

$$= \frac{\pi A^2}{2}\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\}+\frac{N_0}{2}, \omega_0-\omega_B<\omega<\omega_0+\omega_B$$

Problem 18. Consider a Gaussian white noise of zero mean and power spectral density $\frac{N_0}{2}$ applied to a low pass RC filter whose transfer function is $H(f)=\frac{1}{1+i2\pi fRC}$. Find the autocorrelation function.

Solution:

The transfer function of a RC circuit is given. We know if $X(t)$ is the input process and $Y(t)$ is the output process of a linear system, then the relation between their spectral densities is $S_{YY}(\omega)=|H(\omega)|^2 S_{XX}(\omega)$

The given transfer function is in terms of frequency f

$$S_{YY}(f)=|H(f)|^2 S_{XX}(f)$$

$$S_{YY}(f)=\frac{1}{1+4\pi^2 f^2R^2C^2}\frac{N_0}{2}$$

$$\therefore R_{YY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{i\omega\tau}d\,\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i2\pi\tau f}}{1+4\pi^2f^2R^2C^2}\frac{N_0}{2}d\,f$$

$$=\frac{N_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{i(2\pi\tau)f}}{4\pi^2R^2C^2\left(\frac{1}{4\pi^2R^2C^2}+f^2\right)}df$$

$$=\frac{N_0}{16\pi^3R^2C^2}\int_{-\infty}^{\infty}\frac{e^{i(2\pi\tau)f}}{\left(\frac{1}{2\pi RC}\right)^2+f^2}df$$

We know from contour integration $\int_{-\infty}^{\infty}\frac{e^{imx}}{a^2+x^2}dx=\frac{\pi}{a}e^{-|m|a}$

$$\therefore R_{YY}(\tau)=\frac{N_0}{16\pi^3R^2C^2}\frac{\pi}{\frac{1}{2\pi RC}}e^{-\frac{\tau}{2\pi RC}}$$

$$=\frac{N_0}{16\pi^3R^2C^2}2\pi^2RC\,e^{-\frac{\tau}{2\pi RC}}$$

$$=\frac{N_0}{8\pi RC}e^{-\frac{\tau}{2\pi RC}}$$

Problem 19. A wide-sense stationary noise process N(t) has an autocorrelation function $R_{NN}(\tau)=Pe^{-3|\tau|}$, where P is a constant Find its power spectrum

Solution:

Given the autocorrelation function of the noise process $N(t)$ is $R_{NN}(\tau) = Pe^{-3|\tau|}$.

$$\therefore S_{NN}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} P e^{-3|\tau|} e^{-i\omega\tau} d\tau$$

$$= P\left\{ \int_{-\infty}^{0} e^{3\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-3\tau} e^{-i\omega\tau} d\tau \right\}$$

$$= P\left\{ \int_{-\infty}^{0} e^{(3-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(3+i\omega)\tau} d\tau \right\}$$

$$= P\left\{ \left[\frac{e^{(3-i\omega)\tau}}{3-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(3+i\omega)\tau}}{-(3+i\omega)} \right]_{0}^{\infty} \right\}$$

$$= P\left\{ \frac{1}{3-i\omega}(1-0) - \frac{1}{3+i\omega}(0-1) \right\}$$

$$= P\left\{ \frac{1}{3-i\omega} + \frac{1}{3+i\omega} \right)$$

$$= P\left\{ \frac{3+i\omega+3-i\omega}{(3-i\omega)(3+i\omega)} \right\} = \frac{6P}{9+\omega^2}$$

Problem 20. A wide sense stationary process X(t) is the input to a linear system with impulse response $h(t) = 2e^{-7t}, t \geq 0$. If the autocorrelation function of X(t) is $R_{XX}(\tau) = e^{-4|\tau|}$, find the power spectral density of the output process Y(t).

Solution:

Given X(t) is a WSS process which is the input to a linear system and so the output process Y(t) is also a WSS process (by property autocorrelation function)

Further the spectral relationship is $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

Where $S_{XX}(\omega) =$ Fourier transform of $R_{XX}(\tau)$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-4|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{4\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-4\tau} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\tau(4-i\omega)} d\tau + \int_{0}^{\infty} e^{-\tau(4+i\omega)} d\tau$$

$$= \left[\frac{e^{\tau(4-i\omega)}}{4-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-\tau(4+i\omega)}}{4+i\omega} \right]_{0}^{\infty}$$

$$= \frac{1}{4-i\omega}\left[e^{0}-e^{-\infty}\right] + \frac{1}{4+i\omega}\left[e^{-\infty}-e^{0}\right]$$

$$= \frac{1}{4-i\omega} - \frac{1}{4+i\omega}$$

$$S_{XX}(\omega) = \frac{4+i\omega-4+i\omega}{(4-i\omega)(4+i\omega)} = \frac{2i\omega}{16+\omega^2}$$

$H(\omega) =$ Fourier transform of $h(t)$

$$= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt$$

$$= \int_{0}^{\infty} 2e^{-7t}.e^{-i\omega t}dt \qquad \left[\because h(t)=0 \text{ } if \text{ } t<0\right]$$

$$= 2\int_{0}^{\infty} e^{-(7+i\omega)t}dt$$

$$= 2\left[\frac{e^{-(7+i\omega)t}}{-(7+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{-2}{7+i\omega}\left[e^{-\infty}-e^{0}\right] = \frac{2}{7+i\omega}$$

$$\therefore \left|H(\omega)\right| = \frac{2}{\left|7+i\omega\right|} = \frac{2}{\sqrt{49+\omega^2}}$$

$$\therefore \left|H(\omega)\right|^2 = \frac{4}{49+\omega^2}$$

Substituting in (1) we get the power spectral density of Y(t), $S_{YY}(\omega) = \frac{4}{49+\omega^2}.\frac{2i\omega}{16+\omega^2} = \frac{8i\omega}{(49+\omega^2)(16+\omega^2)}$.

Problem 21. A random process X(t) with $R_{XX}(\tau) = e^{-2|\tau|}$ is the input to a linear system whose impulse response is $h(t) = 2e^{-t}, t \ge 0$. Find cross correlation $R_{YY}(\tau)$ between the input process X(t) and the output process Y(t).

Solution:

The cross correlation between input X(t) and output Y(t) to a linear system is

$$R_{XY}(\tau) = R_{XX}(\tau)h(\tau)$$

Taking Fourier transforms, we get

$$S_{XY}(\omega) = R_{XX}(\omega)H(\omega)$$

Given $R_{XX}(\tau) = e^{-2|\tau|}$

$$\therefore \; S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2|\tau|} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-2\tau} e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{(2-i\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(2+i\omega)\tau} d\tau$$

$$= \left[\frac{e^{(2-i\omega)\tau}}{2-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(2+i\omega)\tau}}{-(2+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{2-i\omega}(1-0) - \frac{1}{2+i\omega}(0-1)$$

$$= \frac{1}{2-i\omega} + \frac{1}{2+i\omega} = \frac{4}{4+\omega^2}$$

$$\therefore \; H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t} dt$$

$$= \int_{-\infty}^{\infty} 2e^{-t} \, e^{-i\omega t} dt$$

$$= 2\int_{0}^{\infty} e^{-(1+i\omega)t} \, dt = 2\left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right]_{0}^{\infty}$$

$$= \frac{-2}{1+i\omega}[0-1] = \frac{2}{1+i\omega}$$

$$\therefore S_{XY}(\omega) = \frac{4}{4+\omega^2} \cdot \frac{2}{1+i\omega}$$

$$= \frac{8}{(2+i\omega)(2-i\omega)(1+i\omega)}$$

Let $\dfrac{8}{(2+i\omega)(2-i\omega)(1+i\omega)} = \dfrac{A}{2+i\omega} + \dfrac{B}{2-i\omega} + \dfrac{C}{1+i\omega}$

$\therefore 8 = A(2+i\omega)(1+i\omega) + B(2+i\omega)(1+i\omega) + C(2+i\omega)(2-i\omega)$

Put $\omega = i2$, then $8 = A(4)(-1) \Rightarrow A = -2$

$\omega = i$ then $8 = C(1)(3) \Rightarrow C = \dfrac{8}{3}$

$\omega = -i2$, then $8 = B(4)(3) \Rightarrow B = \dfrac{2}{3}$

$$\therefore \; S_{XY}(\omega) = \frac{-2}{2+i\omega} + \frac{\frac{2}{3}}{2-i\omega} + \frac{8/3}{1+i\omega}$$

Taking inverse Fourier transform

$$\therefore R_{XY}(\tau) = F^{-1}\left(\frac{-2}{2+i\omega}\right) + +F^{-1}\left(\frac{\frac{2}{3}}{2-i\omega}\right) + F^{-1}\left(\frac{8/3}{1+i\omega}\right)$$

$$= -2.e^{-2\tau}\mu(\tau) + \frac{2}{3}e^{2\tau}\mu(-\tau) + \frac{8}{3}e^{-\tau}\mu(\tau)$$

Problem 22. If X(t) is a band limited process such that $S_{XX}(\omega) = 0$,where $|\omega| > \sigma$ prove that $2\left[R_{XX}(0) - R_{XX}(\tau)\right] \le \sigma^2\tau^2 R_{XX}(0)$

Solution:

Given $S_{XX}(\omega) = 0$, $|\omega| > \sigma$

$\Rightarrow S_{XX}(\omega) = 0$ if $\omega < -\sigma$ $or\, \omega > \sigma$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)e^{i\tau\omega}d\omega$$

$$= \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(\cos\tau\omega + i\sin\tau\omega)dw$$

$$= \frac{1}{2\pi}\left\{\int_{-\sigma}^{\sigma} S_{XX}(\omega)\cos\tau\omega\, dw + i\int_{-\sigma}^{\sigma} S_{XX}(\omega)\sin\tau\omega\, dw\right\}$$

$$\therefore \int_{-\sigma}^{\sigma} S_{XX}(\omega)\cos\tau\,\omega\, d\,\omega = 2\int_{0}^{\sigma} S_{XX}(\omega)\cos\tau\,\omega\, d\,\omega \text{ and } \int_{-\sigma}^{\sigma} S_{XX}(\omega)\sin\tau\,\omega\, d\,\omega = 0$$

$$\therefore R_{XX}(\tau) = \frac{1}{2\pi}2\int_{0}^{\sigma} S_{XX}(\omega)\cos\tau\omega d\omega$$

$$= \frac{1}{\pi}\int_{0}^{\sigma} S_{XX}(\omega)\cos\tau\omega d\omega$$

$$\therefore R_{XX}(0) = \frac{1}{\pi}\int_{0}^{\sigma} S_{XX}(\omega)d\omega \quad ...(1)$$

$$\therefore R_{XX}(0) - R_{XX}(\tau) = \frac{1}{\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)d\omega - \frac{1}{\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)\cos\omega\tau d\omega$$

$$= \frac{1}{\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)(1-\cos\omega\tau)d\omega$$

$$= \frac{1}{\pi}\int_{-\sigma}^{\sigma} S_{XX}(\omega)2\sin^2\left(\frac{\omega\tau}{2}\right)d\omega$$

We know that $\sin^2\theta \le \theta^2$

$$\therefore 2\sin^2\left(\frac{\omega\tau}{2}\right)\le\left(\frac{\omega\tau}{2}\right)^2=\frac{\omega^2\tau^2}{2}<\frac{\sigma^2\tau^2}{2}\left[\because 0\le\omega\le\sigma,\omega^2\le\sigma^2\right]$$

$$\therefore R_{XX}(0)-R_{XX}(\tau)\le\frac{1}{\pi}.\int_0^{\sigma}S_{XX}(\omega)\frac{\sigma^2\tau^2}{2}d\omega$$

$$\le\frac{1}{\pi}\frac{\sigma^2\tau^2}{2}\int_0^{\sigma}S_{XX}(\omega)d\omega$$

$$\le\frac{\tau^2\sigma^2}{2\pi}\int_0^{\sigma}S_{XX}(\omega)d\omega$$

$$\le\frac{\tau^2\sigma^2}{2\pi}\int_0^{\sigma}S_{XX}(\omega)d\omega$$

$$\le\frac{\sigma^2\tau^2}{2\pi}R_{XX}(0) \qquad \text{[Using (1)]}$$

$$\therefore 2\left[R_{XX}(0)-R_{XX}(\tau)\right]\le\sigma^2\tau^2R_{XX}(0)$$

Problem 23. The autocorrelation function of the Poisson increment process is given by $R(\tau)=\begin{cases}\lambda^2 & for\ |\tau|>\in\\ \lambda^2+\frac{\lambda}{\in}\left(1-\frac{|\tau|}{\in}\right) & for\ |\tau|\le\in\end{cases}$. Prove that its spectral density is given by

$$S(\omega)=2\pi\lambda^2\delta(\omega)+\frac{4\lambda\sin^2\frac{\omega t}{2}}{\in^2\omega^2}.$$

Solution:

Given the autocorrelation function $R(\tau)=\begin{cases}\lambda^2 & for\ \tau>-\in\ or\ \tau>\in\\ \lambda^2+\frac{\lambda}{\in}\left(1-\frac{|\tau|}{\in}\right) & for-\in\le\tau\le\in\end{cases}$

$\therefore$ The spectral density is given by $S(\omega)=\int_{-\infty}^{\infty}R(\tau)e^{-i\tau\omega}d\tau$, by definition

$$=\int_{-\infty}^{-\in}R(\tau)e^{-i\tau\omega}d\tau+\int_{-\in}^{\in}R(\tau)e^{-i\tau\omega}d\tau+\int_{\omega}^{\infty}R(\tau)e^{-i\tau\omega}d\tau$$

$$=\int_{-\infty}^{-\in}\lambda^2e^{-i\tau\omega}d\tau+\int_{-\in}^{\in}\left[\lambda^2+\frac{\lambda}{\in}\left(1-\frac{|\tau|}{\in}\right)\right]e^{-i\tau\omega}d\tau+\int_{\omega}^{\infty}\lambda^2e^{-i\tau\omega}d\tau$$

$$=\int_{-\infty}^{-\in}\lambda^2e^{-i\tau\omega}d\tau+\int_{-\in}^{\in}\lambda^2e^{-i\tau\omega}d\tau+\int_{\in}^{\infty}\lambda^2e^{-i\tau\omega}d\tau+\int_{-\in}^{\in}\frac{\lambda}{\in}\left(1-\frac{|\tau|}{\in}\right)e^{-i\tau\omega}d\tau$$

$$=\int_{-\infty}^{-\infty}\lambda^2e^{-i\tau\omega}d\tau+\frac{\lambda}{\in}\int_{-\in}^{\in}\left(1-\frac{|\tau|}{\in}\right)e^{-i\tau\omega}d\tau$$

$$= F\left(\lambda^2\right) + \frac{\lambda}{\in} \int_{-\in}^{\in} \left(1 - \frac{|\tau|}{\in}\right)\left(\cos\tau\omega - i\sin\tau\omega\right) d\tau$$

Where $F\left(\lambda^2\right)$ is the Fourier transform of λ^2

$$S(\omega) = F\left(\lambda^2\right) + \frac{\lambda}{\in}\left\{\int_{-\in}^{\in}\left(1 - \frac{|\tau|}{\in}\right)\cos\tau\omega d\tau - i\int_{-\in}^{\in}\left(1 - \frac{|\tau|}{\in}\right)\sin\tau\omega d\tau\right\} \qquad .....(1)$$

But $\left(1 - \frac{|\tau|}{\in}\right)$ $\cos\tau\omega$ is an even function of τ and $\left(1 - \frac{|\tau|}{\in}\right)$ $\sin\tau\omega$ is an odd function of τ

$$\therefore \int_{-\in}^{\in}\left(1 - \frac{|\tau|}{\in}\right)\sin\tau\omega d\tau = 0 \;\; and \;\; \int_{-\in}^{\in}\left(1 - \frac{|\tau|}{\in}\right)\cos\tau\omega d\tau = 2\int_{0}^{\in}\left(1 - \frac{\tau}{\in}\right)\cos\tau\omega d\tau \left(\because \tau > 0; |\tau| = \tau\right)$$

$$S(\omega) = F\left(\lambda^2\right) + \frac{\lambda}{\in} 2\int_{0}^{\in}\left(1 - \frac{\tau}{\in}\right)\cos\tau\omega d\tau$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\in}\left[\left(1 - \frac{\tau}{\in}\right)\left(\frac{\sin\tau\omega}{\omega}\right) - \left(\frac{-1}{\in}\right)\left(\frac{-\cos\tau\omega}{\omega^2}\right)\right]_0^{\in} \text{ [by Bernoulli's formula]}$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\in}\left[\frac{1}{\omega}\left(1 - \frac{\tau}{\in}\right)\sin\tau\omega - \frac{1}{\in\omega^2}\cos\tau\omega\right]_0^{\in}$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\in}\left[0 - \frac{1}{\in\omega^2}\left(\cos\in\omega - \cos 0\right)\right]$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\in}\frac{1}{\in\omega^2}\left[1 - \cos\in\omega\right]$$

$$= F\left(\lambda^2\right) + \frac{2\lambda}{\in^2\omega^2}.2\sin^2\frac{\in\omega}{2}$$

$$S(\omega) = F\left(\lambda^2\right) + \frac{4\lambda}{\in^2\omega^2}\sin^2\frac{\in\omega}{2} \qquad ....(1)$$

To find the value of $F\left(\lambda^2\right)$, we shall find the inverse Fourier transform of $S(\omega)$,

$$R(\tau) = F^{-1}\left(S(\omega)\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

Consider $S(\omega) = 2\pi\lambda^2\delta(\omega)$, where $\delta(\omega)$ is the unit impulse function.

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\lambda^2\delta(\omega) e^{i\omega\tau} d\omega$$

$$= \lambda^2\int_{-\infty}^{\infty}\delta(\omega) e^{i\omega\tau} d\omega$$

$$= \lambda^2.1 \qquad \left[\because \int_{-\infty}^{\infty}\phi(t)\delta(t) dt = \phi(0)\right.$$

$$= \lambda^2 \qquad \Rightarrow \int_{-\infty}^{\infty} e^{i\omega\tau}\delta(\omega)\,d\omega = e^0 = 1 \;\; as \;\; \phi(\omega) = e^{i\tau\omega} \Big]$$

Thus $\lambda^2 = R(\tau)$ Taking Fourier transform

$F(\lambda^2) = F(R(\tau)) = S(\omega) = 2\pi\lambda^2\delta(\omega)$

Substituting in (1) we get $S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda}{\epsilon^2\,\omega^2}\sin^2\left(\frac{\omega\,\epsilon}{2}\right)$.

Problem 24. Suppose X(t) be the input process to a linear system with autocorrelation $R_{XX}(\tau) = 3\delta(\tau)$ and the impulse response $H(\omega) = \frac{1}{6+i\omega}$, then find(i) the autocorrelation of the output process Y(t). (ii) the power spectral density of Y(t).

Solution:

Given $R_{XX}(\tau) = 3\delta(\tau)$ and $H(\omega) = \frac{1}{6+i\omega}$

$$\therefore S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} 3\delta(\tau)e^{-i\omega\tau}d\tau$$

$$= 3\int_{-\infty}^{\infty} \delta(\tau)e^{-i\omega\tau}d\tau$$

We know $\int_{-\infty}^{\infty} \delta(\tau)\phi(\tau) = \phi(0)$

Here $\phi(\tau) = e^{-i\omega\tau}$ $\therefore \phi(0) = 1$

$\therefore S_{XX}(\omega) = 3.1 = 3$

We know the spectral relation between input and output process is $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

But $|H(\omega)|^2 = \frac{1}{36+\omega^2}$

$\therefore S_{YY}(\omega) = \frac{3}{36+\omega^2}$ which is the power spectral density of $Y(t)$

Now the autocorrelation of $Y(t)$ is $R_{YY}(\tau) = F^{-1}(S_{YY}(\omega))$

$$R_{YY}(\tau) = F^{-1}\left(\frac{3}{36+\omega^2}\right)$$

We know $F^{-1}\left(\frac{2\alpha}{\alpha^2+\omega^2}\right) = e^{-\alpha|\tau|}$

$$\therefore R_{YY}(\tau) = \frac{3}{2.6}F^{-1}\left(\frac{2.6}{6^2+\omega^2}\right) \quad [Here\ \alpha = 6]$$

$$=\frac{1}{4}e^{-6|\tau|}$$

(ii) $S_{YY}(\omega)=\int_{-\infty}^{\infty} R_{YY}(\tau)e^{-i\omega\tau}d\tau$

$$=\int_{-\infty}^{\infty}\frac{1}{4}e^{-6|\tau|}e^{-i\tau\omega}d\tau$$

$$=\frac{1}{4}\left\{\int_{-\infty}^{0}e^{(6\tau-i\omega)\tau}d\tau+\int_{0}^{\infty}e^{-(6+i\omega)\tau}d\tau\right\}$$

$$=\frac{1}{4}\left\{\left[\frac{e^{(6-i\omega)\tau}}{6-i\omega}\right]_{-\infty}^{0}+\left[\frac{e^{-(6+i\omega\tau)}}{-(6+i\omega)}\right]_{0}^{\infty}\right\}$$

$$=\frac{1}{4}\left\{\frac{1}{6-i\omega}(1-0)-\frac{1}{6+i\omega}(0-1)\right\}$$

$$=\frac{1}{4}\left\{\frac{1}{6-i\omega}+\frac{1}{6+i\omega}\right\}$$

$$=\frac{1}{4}\left\{\frac{6+i\omega+6-i\omega}{(6-i\omega)(6+i\omega)}\right\}$$

$$=\frac{1}{4}\left\{\frac{12}{36+\omega^2}\right\}=\frac{3}{36+\omega^2}.$$

Problem 25. Show that the power spectrum $S_{YY}(\omega)$ of the output of a linear system with system function $H(\omega)$ is given by $S_{YY}(\omega)=S_{XX}(\omega)|H(\omega)|^2$ where $S_{XX}(\omega)$ is the power spectrum of the input.

Solution:

If $\{X(t)\}$ is a WSS and if $y(t)=\int_{-\infty}^{\infty}h(\alpha)X(t-\alpha)d\alpha$

We shall prove that $S_{YY}(\omega)=S_{XY}(\omega)|H(\omega)|^2$.

Consider $Y(t)=\int_{-\infty}^{\infty}X(t-\alpha)h(\alpha)d\alpha$

$$X(t+\tau)Y(t)=\int_{-\infty}^{\infty}X(t+\tau)X(t-\alpha)h(\alpha)d\alpha$$

$$E\left[X(t+\tau)Y(t)\right]=\int_{-\infty}^{\infty}E\left[X(t+\tau)\,X(t-\alpha)\right]h(\alpha)d\alpha$$

$$R_{YX}(-\tau)=\int_{-\infty}^{\infty}R_{XX}(\tau+\alpha)h(\alpha)d\alpha$$

$$R_{XY}(\tau)=\int_{-\infty}^{\infty}R_{XX}(\tau-\beta)h(-\beta)d\beta$$

$R_{XY}(\tau)=R_{XX}(\tau)*h(-\tau)$

$Y(t)Y(t-\tau)=\int_{-\infty}^{\infty}X(t-\alpha)Y(t-\tau)h(\alpha)d\alpha$

$\therefore E\left[Y(t)Y(t-\tau)\right]=\int_{-\infty}^{\infty}R_{XY}(\tau-\alpha)h(\alpha)d\alpha$

Assuming that $\{X(t)\}$ & $\{Y(t)\}$ are jointly WSS

$R_{YY}(\tau)=R_{XY}(\tau)*h(\tau)$--------- (2)

Taking Fourier transform of (1) we get

$S_{XY}(\omega)=S_{XX}(\omega)\,H^*(\omega)$------ (3)

Where $H^*(\omega)$ is the conjugate of $H(\omega)$

Taking Fourier transform of (2) we get

$S_{YY}(\omega)=S_{XY}(\omega)\,H(\omega)$--------- (4)

Inserting (3) in (4)

$S_{YY}(\omega)=S_{XX}(\omega)\,H^*(\omega)H(\omega)$

$S_{YY}(\omega)=S_{XX}(\omega)\,|H(\omega)|^2$

Problem 26. A system has an impulse response $h(t)=e^{-\beta t}U(t)$, find the power spectral density of the output $Y(t)$ corresponding to the input $X(t)$.

Solution:

Given $h(t)=e^{-\beta t}, t\geq 0$

$H(\omega)=\int_{-\infty}^{\infty}h(t)\,e^{-i\omega t}dt$

$H(\omega)=\int_{0}^{\infty}e^{-\beta t}e^{-i\omega t}dt$

$=\int_{0}^{\infty}e^{-t(\beta+i\omega)}dt$

$=\left[\frac{e^{-t(\beta+i\omega)}}{-(\beta+i\omega)}\right]_0^{\infty}$

$H(\omega)=\frac{1}{\beta+i\omega}$

$H^*(\omega)=\frac{1}{\beta-i\omega}$

$|H(\omega)|^2=H(\omega)H^*(\omega)$

$=\left(\frac{1}{\beta+i\omega}\right)\left(\frac{1}{\beta-i\omega}\right)$

$$=\frac{1}{\beta^2+\alpha^2}$$

$$\therefore S_{YY}(\omega)=|H(\omega)|^2 S_{XX}(\omega)$$

$$S_{YY}(\omega)=\frac{S_{XX}(\omega)}{\beta^2+\alpha^2}$$

Problem 27. If $X(t)$ is the input voltage to a circuit and $Y(t)$ is the output voltage, $\{X(t)\}$ is a stationary random process with $\mu_x=0$ and $R_{XX}(\tau)=e^{-2|\tau|}$. Find $\mu_y, S_{XX}(\omega)$ and $S_{YY}(\omega)$, if the system function is given by $H(\omega)=\frac{1}{\omega^2+2^2}$.

Solution:

Given Mean $\left[X(t)\right]=\mu_X=0$

$$Y(t)=\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)d\alpha$$

$$E\left[Y(t)\right]=\int_{-\infty}^{\infty} h(\alpha)E\left[X(t-\alpha)\right]d\alpha=0$$

$$S_{XX}(\omega)=\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau$$

$$=\int_{-\infty}^{\infty} e^{-2|\tau|}e^{-i\omega\tau}d\tau=\int_{-\infty}^{\infty} e^{-2|\tau|}\cos\omega\tau\, d\tau=2\int_{0}^{\infty} e^{-2\tau}\cos\omega\tau\, d\tau$$

$$S_{XX}(\omega)=2\left[\frac{2}{\omega^2+4}\right]=\frac{4}{\omega^2+4}$$

$$H(\omega)=\frac{1}{\omega^2+4}$$

$$S_{YY}(\omega)=|H(\omega)|^2 S_{XX}(\omega)=\left(\frac{1}{\omega^2+4}\right)\frac{4}{\omega^2+4}$$

$$S_{YY}(\omega)=\frac{4}{\left(\omega^2+4\right)^2}$$

Problem 28. $X(t)$ is the input voltage to a circuit (system) and $Y(t)$ is the output voltage. $\{X(t)\}$ is a stationary random process with $\mu_x=0$ and $R_{XX}(\tau)=e^{-\alpha|\tau|}$. Find μ_y, $S_{yy}(\omega)$ & $R_{yy}(\tau)$ if the power transfer function is $H(\omega)=\frac{R}{R+iL\omega}$

Solution:

$$Y(t)=\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)d\alpha$$

$$E\left[Y(t)\right] = \int_{-\infty}^{\infty} h(\alpha) E\left[X(t-\alpha) d\alpha\right]$$

$$E\left[Y(t)\right] = 0$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\alpha\tau} e^{-i\omega\tau} d\tau + \int_{0}^{\infty} e^{-\alpha\tau} e^{-i\omega\tau} d\tau$$

$$= \left[\frac{e^{(\alpha-i\omega)\tau}}{\alpha - i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(\alpha+i\omega)\tau}}{-(\alpha+i\omega)}\right]_{0}^{\infty} = \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

$$S_{YY}(\omega) = S_{XX}(\omega) \left|H(\omega)\right|^2$$

$$= \frac{2\alpha}{\alpha^2 + \omega^2} \frac{R}{R^2 + L^2\omega^2}$$

Consider,

$$\frac{2\alpha R^2}{\left(\alpha^2 + \omega^2\right)\left(R^2 + L^2\omega^2\right)} = \frac{A}{\alpha^2 + \omega^2} + \frac{B}{R^2 + L^2\omega^2}$$

By partial fractions

$$S_{YY}(\omega) = \frac{2\alpha\left(\frac{R^2}{R^2 - L^2\alpha^2}\right)}{\alpha^2 + \omega^2} + \frac{2\alpha\left(\frac{R^2}{\alpha^2 - \frac{R^2}{L^2}}\right)}{R^2 + L^2\omega^2}$$

$$= \frac{2\alpha\left(\frac{R}{L}\right)^2}{\left(\frac{R}{L}\right)^2 - \alpha^2} \cdot \frac{1}{\alpha^2 + \omega^2} + \frac{2\alpha\left(\frac{R^2}{L^2}\right)^2}{\alpha^2 - \left(\frac{R}{L}\right)^2} \cdot \frac{1}{\left(\frac{R}{L}\right)^2 + \omega^2}$$

$$= \lambda . \frac{1}{\alpha^2 + \omega^2} + \mu . \frac{1}{\left(\frac{R}{L}\right)^2 + \omega^2}$$

$$R_{YY}(\tau) = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\alpha^2 + \omega^2} d\omega + \frac{\mu}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\left(\frac{R}{L}\right)^2 + \omega^2} d\omega$$

By contour integration technique we know that

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{z^2 + b^2} dz = \frac{\pi}{b} e^{-ab},\ a > 0$$

$$\therefore R_{YY}(\tau)=\frac{\left(\frac{R}{L}\right)^2}{\left(\frac{R}{L}\right)^2-\alpha^2}e^{-\alpha|\tau|}+\frac{\left(\frac{R}{L}\right)^2\alpha}{\alpha^2-\left(\frac{R}{L}\right)^2}e^{-\left(\frac{R}{L}\right)|\tau|}$$

.

Problem 29. $X(t)$ is the i/P voltage to a circuit and $Y(t)$ is the O/P voltage. $X(t)$ is a stationary random process with zero mean and autocorrelation $R_{XX}(\tau)=e^{-2|\tau|}$. Find the mean of $Y(t)$ and its PSD if the system function $H(\omega)=\frac{1}{J\omega+2}$.

Solution:

$$H(\omega)=\frac{1}{J\omega+2}$$

$$\Rightarrow H(0)=\frac{1}{2}$$

$$E\left[Y(t)\right]=E\left[X(t)\right].H(0)=0$$

$$\left|H(\omega)\right|^2=\frac{1}{\omega^2+4}$$

$$S_{XX}(\omega)=F\left[R_{XX}(\tau)\right]=F\left[e^{-2|\tau|}\right]=\frac{4}{\omega^2+4}$$

$$S_{YY}(\omega)=\left|H(\omega)\right|^2 S_{XX}(\omega)=\frac{4}{\left(\omega^2+4\right)^2}$$

$$\therefore E\left(Y^2\right)=\int_{-2}^{0}(2+\tau)\left(9+2e^{\tau}\right)d\tau+\int_{0}^{2}(2-\tau)\left(9+2e^{-\tau}\right)d\tau$$

$$=\left[18\tau+4e^{\tau}+\frac{9\tau^2}{2}+2e^{\tau}(\tau-1)\right]_{-2}^{0}+\left[18\tau-4e^{\tau}-\frac{9\tau^2}{2}2e^{-\tau}(\tau+1)\right]_{0}^{2}$$

$$\therefore E\left(Y_2\right)=40.542$$

$$V\text{ar}(Y)=E\left(Y^2\right)-\left[E(Y)\right]^2$$

$$=40.542-36=4.542$$

Problem 30. Consider a system with transfer function $\frac{1}{1+i\omega}$. An input signal with autocorrelation function $m\delta(\tau)+m^2$ is fed as input to the system. Find the mean and mean-square value of the output.

Solution:

Given, $H(\omega)=\frac{1}{1+i\omega}$ and $R_{XX}(\tau)=m\delta(\tau)+m^2$

$$S_{XX}(\omega)=m+2\pi m^2\delta(\omega)$$

We know that, $S_{YY}(\omega)=\left|H(\omega)\right|^2 S_{XX}(\omega)$

$$= \left|\frac{1}{1+i\omega}\right|^2 \left[m + 2\pi m^2 \delta(\omega)\right]$$

$$= \frac{1}{1+\omega^2}\left[m + 2\pi m^2 \delta(\omega)\right]$$

$R_{YY}(\tau)$ is the Fourier inverse transform of $S_{YY}(\omega)$.

So, $R_{YY}(\tau) = \frac{m}{2}e^{-|\tau|} + m^2$

We know that $\lim_{\tau \to \infty} R_{XX}(\tau) = \overline{X}^2$

So $\overline{X}^2 = M^2$

$\overline{X} = m$

Also $H(0) = 1$

We know that $\overline{Y} = 1, m = m$

Mean-square value of the output $= \overline{Y}^2 = R_{YY}(0) = \frac{m}{2} + m^2$

Problem 31. If the input to a time-invariant, stable, linear system is a WSS process, prove that the output will also be a WSS process.

Solution:

Let $X(t)$ be a WSS process for a linear time variant stable system with $Y(t)$ as the output process.

Then $Y(t) = \int_{-\infty}^{\infty} h(u) X(t-u)\, du$ where h(t) is weighting function or unit impulse response.

$$\therefore E\left[Y(t)\right] = \int_{-\infty}^{\infty} E\left[h(u) X(t-u)\right] du$$

$$= \int_{-\infty}^{\infty} h(u) E\left[X(t-u)\right] du$$

Since $X(t)$ is a WSS process, $E\left[X(t)\right]$ is a constant μ_X for any t.

$$\therefore E\left[X(t-u)\right] = \mu_X$$

$$\therefore E\left[Y(t)\right] = \int_{-\infty}^{\infty} h(u)\, \mu_X\, du$$

$$= \mu_X \int_{-\infty}^{\infty} h(u)\, du$$

Since the system is stable, $\int_{-\infty}^{\infty} h(u)\, du$ is finite

$\therefore E\left[Y(t)\right]$ is a constant.

Now $R_{YY}(t,t+\tau)=E\left[Y(t)Y(t+\tau)\right]$

$$=E\left[\int_{-\infty}^{\infty}h(u_1)X(t-u_1)du_1\int_{-\infty}^{\infty}h(u_2)X(t+\tau-u_2)du_2\right]$$

$$=E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u_1)h(u_2)X(t-u_1)X(t+\tau-u_2)du_1u_2\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u_1)h(u_2)E\left[X(t-u_1)X(t+\tau-u_2)\right]du_1u_2$$

Since $X(t)$ is a WSS process, auto correlation function is only a function time difference.

$$\therefore R_{YY}(t,t+\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(u_1)h(u_2)R_{XX}(\tau+u_1-u_2)du_1\,du_2$$

When this double integral is evaluated by integrating w.r. to u_1, u_2, the R.H.S is only a function of τ.

$\therefore R_{YY}(t,t+\tau)$ is only a function of time difference τ.

Hence $Y(t)$ is a WSS process.

Problem 32. Let $X(t)$ be a WSS and if $Y(t)=\int_{-\infty}^{\infty}h(u)X(t-u)du$ then show that

a) $R_{XY}(\tau)=h(\tau)*R_{XX}(\tau)$

b) $R_{YX}(\tau)=h(-\tau)*R_{XX}(\tau)$

c) $R_{yy}(\tau)=h(\tau)*R_{xy}(\tau)$

Where $*$ denotes the convolution and $H^*(\omega)$ is the complex conjugate of $H(\omega)$.

Solution:

Given $X(t)$ is WSS $\therefore E\left[X(t)\right]$ is constant and

$R_{XX}(t,t+\tau)=R_{XX}(\tau)$

$$Y(t)=\int_{-\infty}^{\infty}h(u)X(t-u)du$$

Now $R_{XY}(t,t+\tau)=E\left[X(t)Y(t+\tau)\right]$

$$=E\left[X(t)\int_{-\infty}^{\infty}h(u).X(t+\tau-u)du\right]$$

$$=E\left[\int_{-\infty}^{\infty}h(u)X(t)X(t+\tau-u)du\right]=\int_{-\infty}^{\infty}h(u).E\left[X(t)X(t+\tau-u)\right]du$$

Since $X(t)$ is a WSS Process,

$E\left[X(t)X(t+\tau-u)\right]=R_{XX}(\tau-u)$

$$\therefore R_{XY}(t,t+\tau)=\int_{-\infty}^{\infty}h(u)R_{XX}(\tau-u)du$$

$\Rightarrow R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$

(b). Now $R_{YX}(\tau) = R_{XY}(-\tau)$

$= R_{XY}(-\tau) * h(-\tau) \qquad [from\ \ (i)]$

$= R_{XX}(\tau) * h(-\tau) \qquad$ [Since $R_{XX}(\tau)$ is an even function of τ]

(c). $R_{YY}(t, t-\tau) = E[Y(t)Y(t-\tau)]$

$$= E\left[\int_{-\infty}^{\infty} h(u)\ X(t-u)\,du\,Y(t-\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} X(t-u)Y(t-\tau)h(u)\,du\right]$$

$$= \int_{-\infty}^{\infty} E[X(t-u)Y(t-\tau)]h(u)\,du = \int_{-\infty}^{\infty} R_{XY}(\tau-u)h(u)\,du$$

It is a function of τ only and it is true for any τ .

$\therefore R_{YY}(\tau) = R_{XY}(\tau) * h(\tau)$

Problem 33. Prove that the mean of the output of a linear system is given by $\mu_Y = H(0)\mu_X$, where $X(t)$ is WSS.

Solution:

We know that the input $X(t)$, output $Y(t)$ relationship of a linear system can expressed as a convolution $Y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(u)(t-u)\,du$$

Where $h(t)$ is the unit impulse response of the system.

$\therefore$ the mean of the output is

$$E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(u)X(t-u)\,du\right] = \int_{-\infty}^{\infty} h(u)E[X(t-u)]\,du$$

Since X(t) is WSS, $E[X(t)] = \mu_X$ is a constant for any t.

$E[X(t-u)] = \mu_X$

$$\therefore E[Y(t)] = \int_{-\infty}^{\infty} h(u)\mu_X\,du = \mu_X \int_{-\infty}^{\infty} h(u)\,du$$

We know $H(\omega)$ is the Fourier transform of $h(t)$.

i.e. $H(\omega) = \int_{-\infty}^{\infty} h(t)\,dt$

put $\omega = 0$ $\therefore H(0) = \int_{-\infty}^{\infty} h(t)\,dt = \int_{-\infty}^{\infty} h(u)\,du$

$\therefore E[Y(t)] = \mu_X H(0)$.