MISRIMAL NAVAJEE MUNOTH JAIN ENGINEERING COLLEGE, CHENNAI
DEPARTMENT OF MATHEMATICS
PROBABILITY AND RANDOM PROCESSES (MA2261)
SEMESTER –IV
UNIT-III: CLASSIFICATION OF RANDOM PROCESSES
QUESTION BANK ANSWERS
PART-A

Problem 1. Define I & II order stationary Process
Solution:
I Order Stationary Process:
A random process is said to be stationary to order one if is first order density function does not change with a shift in time origin.

i.e., $f_X(x_1 : t_1) = f_X(x_1, t_1 + \delta)$ for any time t_1 and any real number δ.

i.e., $E[X(t)] = \overline{X} =$ Constant.

II Order Stationary Process:
A random process is said to be stationary to order two if its second-order density functions does not change with a shift in time origin.

i.e., $f_X(x_1, x_2 : t_1, t_2) = f_X(x_1, x_2 : t_1 + \delta, t_2 + \delta)$ for all t_1, t_2 and δ.

Problem 2. Define wide-sense stationary process
Solution:
A random process $X(t)$ is said to be wide sense stationary (WSS) process if the following conditions are satisfied

(i). $E[X(t)] = \mu$ i.e., mean is a constant

(ii). $R(\tau) = E[X(t_1)X(t_2)]$ i.e., autocorrelation function depends only on the time difference.

Problem 3. Define a strict sense stationary process with an example
Solution:
A random process is called a strongly stationary process (SSS) or strict sense stationary if all its statistical properties are invariant to a shift of time origin.
This means that $X(t)$ and $X(t+\tau)$ have the same statistics for any τ and any t
Example: Bernoulli process is a SSS process

Problem 4. Define n^{th} order stationary process, when will it become a SSS process?
Solution:
A random process $X(t)$ is said to be stationary to order n or n^{th} order stationary if its n^{th} order density function is invariant to a shift of time origin.

i.e., $f_X(x_1, x_2, ..., x_n, t_1, t_2, ..., t_n) = f_X(x_1, x_2, ..., x_n, t_1 + \delta, t_2 + \delta, ..., t_n + \delta)$ for all $t_1, t_2, ..., t_n$ & h.

A n^{th} order stationary process becomes a SSS process when $n \to \infty$.

Problem 5. When are two random process said to be orthogonal?
Solution:

Two process $\{X(t)\}$ & $\{Y(t)\}$ are said to be orthogonal, if $E\{X(t_1)Y(t_2)\} = 0.$

Problem 6. When are the process $\{X(t)\}$ & $\{Y(t)\}$ said to be jointly stationary in the wide sense?
Solution;

Two random process $\{X(t)\}$ & $\{Y(t)\}$ are said to be jointly stationary in the wide sense, if each process is individually a WSS process and $R_{XY}(t_1, t_2)$ is a function of $(t_2 - t_1)$ only.

7. Write the postulates of a poisson process?
Solution:

If $\{X(t)\}$ represents the number of occurrences of a certain event in $(0, t)$ then the discrete random process $\{X(t)\}$ is called the poisson process, provided the following postualates are satisfied

(i) $P\left[1 \ occumence \ in \ (t, t+\Delta t)\right] = \lambda \Delta t + o(\Delta t)$

(ii) $P\left[no \ occurrence \ in \ (t, t+\Delta t)\right] = 1 - \lambda \Delta t + o(\Delta t)$

(iii) $P\left[2 \ or \ more \ occurrences \ in \ (t, t+\Delta t)\right] = \lambda \Delta t + o(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0, t)$.

(v) The probability that the event occurs in a specified number of times $(t_0, t_0 + t)$ depends only on t, but not on t_0.

8. When is a poisson process said to be homogenous?
Solution:

The rate of occurrence of the event λ is a constant, then the process is called a homogenous poisson process.

9. If the customers arrive at a bank according to a poisson process with a mean rate of 2 per minute, find the probability that, during an 1- minute interval no customer arrives.
Solution:

Here $\lambda = 2$, $t = 1$

$$\therefore P\{X(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \ n = 0, 1, 2, ...$$

Probability during 1-min interval, no customer arrives $= P\{X(t) = 0\} = e^{-2}$.

10. Define ergodic process.
Solution:
A random process $\{X(t)\}$ is said to be ergodic, if its ensemble average are equal to appropriate time averages.

11. Define a Gaussian process.
Solution:
A real valued random process $\{X(t)\}$ is called a Gaussian process or normal process, if the random variables $X(t_1), X(t_2), ..., X(t_n)$ are jointly normal for every $n = 1, 2, ...$ and for any set of $t_1, t_2, ...$

The n^{th} order density of a Gaussian process is given by

$$f(x_1, x_2, ..., x_n; t_1, t_2, ..., t_n) = \frac{1}{(2\pi)^{n/2} |\Lambda|^{1/2}} \exp\left[-\frac{1}{2|\Lambda|} \sum_{i=1}^{n} \sum_{j=1}^{n} |\Lambda|_{ij} (x_i - \mu_i)(x_j - \mu_j)\right]$$

Where $\mu_i = E\{X(t_i)\}$ and Λ is the n^{th} order square matrix(λ_{ij}), where $\lambda_{ij} = C\{X(t_i), X(t_j)\}$ and $|\Lambda|_{ij}$ =Cofactor of λ_{ij} in $|\Lambda|$.

12. Define a Markov process with an example.
Solution:
If for $t_1 < t_2 < t_3 < ... < t_n < t$,

$$P\{X(t) \le x / X(t_1) = x_1, X(t_2) = x_2, ..., X(t_n) = X_n\} = P\{X(t) \le x / X(t_n) = x_n\}$$

then the process $\{X(t)\}$ is called a markov process.
Example: The Poisson process is a Markov Process.

13. Define a Markov chain and give an example.
Solution:
If for all n ,

$$P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ...X_0 = a_0\} = P\{X_n = a_n / X_{n-1} = a_{n-1}\},$$

then the process $\{x_n\}$, $n = 0, 1, ...$ is called a Markov chain.
Example: Poisson Process is a continuous time Markov chain.

Problem 14. What is a stochastic matrix? When is it said to be regular?
Solution: A sequence matrix, in which the sum of all the elements of each row is 1, is called a stochastic matrix. A stochastic matrix P is said to be regular if all the entries of P^m (for some positive integer m) are positive.

Problem 15. If the transition probability matrix of a markov chain is $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ find the steady-state distribution of the chain.

Solution:

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution on stationary state distribution of the markov chain.

By the property of π, $\pi P = \pi$

i.e., $(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$

$\frac{1}{2}\pi_2 = \pi_1$ ----------- (1)

$\pi_1 + \frac{1}{2}\pi_2 = \pi_2$ ------ (2)

Equation (1) & (2) are one and the same.

Consider (1) or (2) with $\pi_1 + \pi_2 = 1$, since π is a probability distribution.

$\pi_1 + \pi_2 = 1$

Using (1), $\frac{1}{2}\pi_2 + \pi_2 = 1$

$\frac{3\pi_2}{2} = 1$

$\pi_2 = \frac{2}{3}$

$\pi_1 = 1 - \pi_2 = 1 - \frac{2}{3} = \frac{1}{3}$

$\pi_2 = 1 - \pi_1 = 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$.

PART-B

Problem 16. a). Define a random (stochastic) process. Explain the classification of random process. Give an example to each class.

Solution:

RANDOM PROCESS

A random process is a collection (orensemble) of random variables $\{X(s,t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (Parameter set or index set).

CLASSIFICATION OF RANDOM PROCESS

Depending on the continuous on discrete nature of the state space S and parameter set T, a random process can be classified into four types:

(i). It both T & S are discrete, the random process is called a discrete random sequence.

Example: If X_n represents the outcome of the n^{th} toss of a fair dice, then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1,2,3,...\}$ and $S = \{1,2,3,4,5,6\}$.

(ii). If T is discrete and S is continuous, the random process is called a continuous random sequence.
Example: If X_n represents the temperature at the end n^{th} hour of a day, then $\{X_n, 1 \le n \le 24\}$ is a continuous random sequence since temperature can take any value is an interval and hence continuous.
(iii). If T is continuous and S is discrete, the random process is called a discrete random process.
Example: If $X(t)$ represents the number of telephone calls received in the interval $(0,t)$ then $\{X(t)\}$ random process, since $S = \{0,1,2,3,...\}$.
(iv). If both T and S are continuous, the random process is called a continuous random process f
Example: If $X(t)$ represents the maximum temperature at a place in the interval $(0,t)$ $\{X(t)\}$ is a continuous random process.

b). Consider the random process $X(t) = \cos(t+\phi)$, where ϕ is uniformly distributed in the interval $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Check whether the process is stationary or not.

Solution:

Since $\emptyset$ is uniformly distributed in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$f(\emptyset) = \frac{1}{\pi}, -\frac{\pi}{2} < \emptyset < \frac{\pi}{2}$$

$$E[X(t)] = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} X(t) f(\emptyset) d\emptyset$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(t+\emptyset).\frac{1}{\pi} d\emptyset$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos(t+\emptyset) d\emptyset$$

$$= \frac{1}{\pi}\left[sin(t+\emptyset)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi} cost \ne \text{Constant.}$$

Since $E[X(t)]$ is a function of t, the random process $\{X(t)\}$ is not a stationary process.

Problem 17. a). Show that the process $\{X(t)\}$ whose probability distribution under certain conditions is given by $P\{X(t)=n\}=\begin{cases}\dfrac{(at)^{n-1}}{(1+at)^{n-1}}, & n=1,2,\ldots\\ \dfrac{at}{1+at}, & n=0\end{cases}$ is evolutionary.

Solution:
The probability distribution is given by

$X(t)=n$	:	0	1	2	3	. . . .
$P(X(t)=n)$	:	$\dfrac{at}{1+at}$	$\dfrac{1}{(1+at)^2}$	$\dfrac{at}{(1+at)^3}$	$\dfrac{(at)^2}{(1+at)^4}$	. . . .

$$E[X(t)]=\sum_{n=0}^{\infty}np_n$$

$$=\frac{1}{(1+at)^2}+\frac{2at}{(1+at)^3}+\frac{3(at)^2}{(1+at)^4}+\ldots$$

$$=\frac{1}{(1+at)^2}\left\{1+2\left(\frac{at}{1+at}\right)+3\left(\frac{at}{1+at}\right)^2+\ldots\right\}$$

$$=\frac{1}{(1+at)^2}\left(1-\frac{at}{1+at}\right)^{-2}$$

$$E[X(t)]=1=\text{Constant}$$

$$E\left[X^2(t)\right]=\sum_{n=0}^{\infty}n^2p_n$$

$$=\sum_{n=1}^{\infty}n^2\frac{(at)^{n-1}}{(1+at)^{n+1}}=\sum_{n=1}^{\infty}\left[n(n+1)-n\right]\frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$=\frac{1}{(1+at)^2}\left[\sum_{n=1}^{\infty}n(n+1)\left(\frac{at}{1+at}\right)^{n-1}-\sum_{n=1}^{\infty}n\left(\frac{at}{1+at}\right)^{n-1}\right]$$

$$=\frac{1}{(1+at)^2}\left[2\left[1-\frac{at}{1+at}\right]^{-3}-\left[1-\frac{at}{1+at}\right]^{-2}\right]$$

$$E\left[X^2(t)\right]=1+2at\neq\text{Constant}$$

$$Var\{X(t)\}=E\left[X^2(t)\right]-\left[E(X(t))\right]^2$$

$$Var\{X(t)\}=2at$$

$\therefore$ The given process $\{X(t)\}$ is evolutionary b). Examine whether the Poisson process $\{X(t)\}$ given by the probability law $P\{X(t)=n\}=\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}$, $n=0,1,2,...$ is evolutionary.

Solution:

$$E\left[X(t)\right]=\sum_{n=0}^{\infty} np_n$$

$$=\sum_{n=0}^{\infty} n\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$=\sum_{n=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{(n-1)!}$$

$$=(\lambda t)e^{-\lambda t}\sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$=(\lambda t)e^{-\lambda t}\left[1+\frac{\lambda t}{1!}+\frac{(\lambda t)^2}{2!}+...\right]$$

$$=(\lambda t)e^{-\lambda t}e^{\lambda t}$$

$$E\left[X(t)\right]=\lambda t$$

$E\left[X(t)\right]\neq$ Constant.

Hence the Poisson process $\{X(t)\}$ is evolutionary.

Problem 18. a). Show that the random process $X(t)=A\cos(\omega \mathrm{t}+\theta)$ is WSS if A & ω are constants and θ is uniformly distributed random variable in $(0,2\pi)$.

Solution:

Since θ is uniformly distributed random variable in $(0,2\pi)$

$$f(\theta)=\begin{cases}\dfrac{1}{2\pi}, & 0<0<2\pi \\ 0 & , elsewhere\end{cases}$$

$$E\left[X(t)\right]=\int_0^{2\pi} X(t)f(\theta)d\theta$$

$$=\int_0^{2\pi}\frac{1}{2\pi}A\,cos(\omega t+\theta)d\theta$$

$$=\frac{A}{2\pi}\int_0^{2\pi} cos(\omega t+\theta)d\theta$$

$$=\frac{A}{2\pi}\left[sin(\omega t+\theta)\right]_0^{2\pi}$$

$$= \frac{A}{2\pi}\left[sin\left(\omega t + 2\pi\right) - sin\left(\omega t\right)\right]$$

$$= \frac{A}{2\pi}\left[sin\left(\omega t\right) - sin\left(\omega t\right)\right] \qquad \left[\because sin\left(2\pi + \theta\right) = sin\theta\right]$$

$$E\left[X\left(t\right)\right] = 0$$

$$R_{XX}\left(t_1, t_2\right) = E\left[X\left(t_1\right)X\left(t_2\right)\right]$$

$$= E\left[A^2 cos\left(\omega t_1 + \theta\right)cos\left(\omega t_2 + \theta\right)\right]$$

$$= A^2 E\left[\frac{cos\left(\omega\left(t_1 + t_2\right) + 2\theta\right) + cos\left(\omega\left(t_1 - t_2\right)\right)}{2}\right]$$

$$= \frac{A^2}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[cos\left(\omega\left(t_1 + t_2\right) + 2\theta\right) + cos\left(\omega\left(t_1 - t_2\right)\right)\right]d\theta$$

$$= \frac{A^2}{4\pi}\left[\frac{sin\left[\omega\left(t_1 + t_2\right) + 2\theta\right]}{2} + \theta cos\omega t\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[2\pi cos\omega t\right]$$

$$= \frac{A^2}{2}cos\omega t = \text{a function of time difference}$$

Since $E\left[X\left(t\right)\right] = \text{constant}$

$R_{XX}\left(t_1, t_2\right) = \text{a function of time difference}$

$\therefore \left\{X\left(t\right)\right\}$ is a WSS.

b). Given a random variable y with characteristic function $\phi\left(\omega\right) = E\left(e^{i\omega y}\right)$ and a random process define by $X\left(t\right) = \cos\left(\lambda t + y\right)$, show that $\left\{X\left(t\right)\right\}$ is stationary in the wide sense if $\phi\left(1\right) = \phi\left(2\right) = 0$.

Solution:

Given $\phi\left(1\right) = 0$

$\Rightarrow E\left[cosy + isiny\right] = 0$

$\therefore E\left[cosy\right] = E\left[\sin y\right] = 0$

Also $\phi\left(2\right) = 0$

$\Rightarrow E\left[cos2y + isin2y\right] = 0$

$\therefore E\left[cos2y\right] = E\left[\sin 2y\right] = 0$

$$E\left\{X\left(t\right)\right\} = E\left[cos\left(\lambda t + y\right)\right]$$

$$= E\left[cos\lambda t cosy - sin\lambda t siny\right]$$

$$= cos\lambda t\, E[cos\lambda t] - sin\lambda t\ E[siny] = 0$$

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$$
$$= E[cos(\lambda t_1 + y)cos(\lambda t_2 + y)]$$
$$= E\left[\frac{cos(\lambda(t_1+t_2)+2y)+cos(\lambda(t_1-t_2))}{2}\right]$$
$$= \frac{1}{2}E[cos(\lambda(t_1+t_2)+2y)+cos(\lambda(t_1-t_2))]$$
$$= \frac{1}{2}E[cos\lambda(t_1+t_2)cos2y - sin\lambda(t_1+t_2)sin2y + cos(\lambda(t_1-t_2))]$$
$$= \frac{1}{2}cos\lambda(t_1+t_2)E(cos2y) - \frac{1}{2}sin\lambda(t_1+t_2)E(sin2y) + \frac{1}{2}cos(\lambda(t_1-t_2))$$
$$= \frac{1}{2}cos(\lambda(t_1-t_2)) = \text{a function of time difference.}$$

Since $E[X(t)] = \text{constant}$

$R_{XX}(t_1,t_2) = \text{a function of time difference}$

$\therefore \{X(t)\}$ is stationary in the wide sense .

Problem 19. a). If a random process $\{X(t)\}$ is defined by $\{X(t)\} = \sin(\omega\,\mathrm{t} + Y)$ where Y is uniformly distributed in $(0, 2\pi)$. Show that $\{X(t)\}$ is WSS.

Solution:

Since y is uniformly distributed in $(0, 2\pi)$,

$$f(y) = \frac{1}{2\pi},\ 0 < y < 2\pi$$

$$E[X(t)] = \int_0^{2\pi} X(t)f(y)dy$$
$$= \frac{1}{2\pi}\int_0^{2\pi} sin(\omega t + y)dy$$
$$= \frac{1}{2\pi}\left[-cos(\omega t + y)\right]_0^{2\pi}$$
$$= -\frac{1}{2\pi}[cos(\omega t + 2\pi) - cos\omega t] = 0$$

$$R_{XX}(t_1,t_2) = E[sin(\omega t_1 + y)sin(\omega t_2 + y)]$$
$$= E\left[\frac{cos(\omega(t_1-t_2)) - cos(\omega(t_1+t_2)+2y)}{2}\right]$$
$$= \frac{1}{2}E[cos(\omega(t_1-t_2))] - \frac{1}{2}E[cos(\omega(t_1+t_2)+2y)]$$

$$= \frac{1}{2} cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{2}\int_0^{2\pi} cos\left(\omega\left(t_1 + t_2\right) + 2y\right)\frac{1}{2\pi}dy$$
$$= \frac{1}{2} cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{4\pi}\left[\frac{sin\left(\omega\left(t_1 + t_2\right) + 2y\right)}{2}\right]_0^{2\pi}$$
$$= \frac{1}{2} cos\left(\omega\left(t_1 - t_2\right)\right) - \frac{1}{8\pi}\left[sin\left(\omega\left(t_1 + t_2\right) + 2\pi\right) - sin\omega\left(t_1 + t_2\right)\right]$$
$$= \frac{1}{2} cos\left(\omega\left(t_1 - t_2\right)\right) \text{is a function of time difference.}$$

$\therefore \left\{X\left(t\right)\right\}$ is WSS.

b). Verify whether the sine wave random process $X\left(t\right) = Y\sin\omega t,\ \ Y$ is uniformly distributed in the interval $\left(-1,1\right)$ is WSS or not

Solution:

Since y is uniformly distributed in $\left(-1,1\right)$,

$$f\left(y\right) = \frac{1}{2},\ -1 < y < 1$$

$$E\left[X(t)\right] = \int_{-1}^{1} X(t)f(y)dy$$
$$= \int_{-1}^{1} y\, sin\omega t \frac{1}{2}dy$$
$$= \frac{sin\omega t}{2}\int_{-1}^{1} ydy$$
$$= \frac{Sin\omega t}{2}\left(0\right) = 0$$

$$R_{XX}\left(t_1, t_2\right) = E\left[X\left(t_1\right)X\left(t_2\right)\right]$$
$$= E\left[y^2 sin\omega t_1 sin\omega t_2\right]$$
$$= E\left[y^2\,\frac{cos\omega\left(t_1 - t_2\right) - cosw\left(t_1 + t_2\right)}{2}\right]$$
$$= \frac{cos\omega\left(t_1 - t_2\right) - cosw\left(t_1 + t_2\right)}{2}E\left(y^2\right)$$
$$= \frac{cos\omega\left(t_1 - t_2\right) - cosw\left(t_1 + t_2\right)}{2}\int_{-1}^{1} y^2 f\left(y\right)dy$$
$$= \frac{cos\omega\left(t_1 - t_2\right) - cosw\left(t_1 + t_2\right)}{2}\frac{1}{2}\int_{-1}^{1} y^2 dy$$

$$= \frac{cos\omega(t_1 - t_2) - cosw(t_1 + t_2)}{4}\left(\frac{y^3}{3}\right)_{-1}^{1}$$

$$= \frac{cos\omega(t_1 - t_2) - cosw(t_1 + t_2)}{4}\left(\frac{2}{3}\right)$$

$$= \frac{cos\omega(t_1 - t_2) - cosw(t_1 + t_2)}{6}$$

$R_{XX}(t_1, t_2) \neq$ a function of time difference alone.
Hence it is not a WSS Process.

Problem 20. a). Show that the process $X(t) = A\cos\lambda t + B\sin\lambda t$ (where A & B are random variables) is WSS, if (i) $E(A) = E(B) = 0$ (ii) $E(A^2) = E(B^2)$ and (iii) $E(AB) = 0$.

Solution:

Given $X(t) = Acos\lambda t + Bsin\lambda t$, $E(A) = E(B) = 0$, $E(AB) = 0$,
$E(A^2) = E(B^2) = k(say)$

$E[X(t)] = cos\lambda t\, E(A) + sin\lambda t\, E(B)$

$E[X(t)] = 0 =$ is a constant. $\because E(A) = E(B) = 0$

$$R(t_1, t_2) = E\{X(t_1)X(t_2)\}$$
$$= E\{(Acos\lambda t_1 + Bsin\lambda t_1)(Acos\lambda t_2 + Bsin\lambda t_2)\}$$
$$= E(A^2)cos\lambda t_1 cos\lambda t_2 + E(B^2)sin\lambda t_1 sin\lambda t_2 + E(AB)[sin\lambda t_1 cos\lambda t_2 + cos\lambda t_1 sin\lambda t_2]$$
$$= E(A^2)cos\lambda t_1 cos\lambda t_2 + E(B^2)sin\lambda t_1 sin\lambda t_2 + E(AB)sin\lambda(t_1 + t_2)$$
$$= k(cos\lambda t_1 cos\lambda t_2 + sin\lambda t_1 sin\lambda t_2)$$
$$= kcos\lambda(t_1 - t_2) = \text{is a function of time difference.}$$

$\therefore \{X(t)\}$ is WSS.

b). If $X(t) = Ycost + Zsint$ for all t & where Y & Z are independent binary random variables. Each of which assumes the values – 1 & 2 with probabilities $\frac{2}{3}$ & $\frac{1}{3}$ respectively, prove that $\{X(t)\}$ is WSS.

Solution:

Given

$Y = y$	:	-1	2
$P(Y = y)$	:	$\frac{2}{3}$	$\frac{1}{3}$

$E(Y)=E(Z)=-1\times\frac{2}{3}+2\times\frac{1}{3}=0$

$E(Y^2)=E(Z^2)=(-1)^2\times\frac{2}{3}+(2)^2\times\frac{1}{3}$

$E(Y^2)=E(Z^2)=\frac{2}{3}+\frac{4}{3}=\frac{6}{3}=2$

Since Y & Z are independent

$E(YZ)=E(Y)E(Z)=0$ ---- (1)

Hence $E[X(t)]=E[ycost+zsint]$

$=E[y]cost+E[z]sint$

$E[X(t)]=0=$ is a constant. $\qquad [\because E(y)=E(z)=0]$

$R_{XX}(t_1,t_2)=E[X(t_1)X(t_2)]$

$=E[(ycost_1+zsint_1)(ycost_2+zsint_2)]$

$=E[y^2cost_1cost_2+yzcost_1sint_2+zysint_1cost_2+z^2sint_1sint_2]$

$=E(y^2)cost_1cost_2+E[yz]cost_1sint_2+E[zy]sint_1cost_2+E[z^2]sint_1sint_2$

$=E(y^2)cost_1cost_2+E(z^2)sint_1sint_2$

$=2[cost_1cost_2+sint_1sint_2]\ [\because E(y^2)=E(z^2)=2]$

$=2cos(t_1-t_2)=$ is a function of time difference.

$\therefore \{X(t)\}$ is WSS.

Problem 21. a). Check whether the two random process given by $X(t)=A\cos\omega t+B\sin\omega t$ & $Y(t)=B\cos\omega t-A\sin\omega t$. Show that $X(t)$ & $Y(t)$ are jointly WSS if A & B are uncorrelated random variables with zero mean and equal variance random variables are jointly WSS.

Solution:

Given $E(A)=E(B)=0$

$Var(A)=Var(B)=\sigma^2$

$\therefore E(A^2)=E(B^2)=\sigma^2$

As $A\&B$ uncorrelated are $E(AB)=E(A)E(B)=0$.

$E[X(t)]=E[Acos\omega t+Bsin\omega t]$

$=E(A)cos\omega t+E(B)sin\omega t=0$

$E[X(t)]=0=$ is a constant.

$R_{XX}(t_1,t_2)=E[X(t_1)X(t_2)]$

$=E[(Acos\omega t_1+Bsin\omega t_2)(Acos\omega t_2+Bsin\omega t_2)]$

$$= E\left[A^2 cos\omega t_1 cos\omega t_2 + ABcos\omega t_1 sin\omega t_2 + BAsin\omega t_1 cos\omega t_2 + B^2 sin\omega t_1 sin\omega t_2\right]$$
$$= cos\omega t_1 cos\omega t_2 E\left[A^2\right] + cos\omega t_1 sin\omega t_2 E\left[AB\right] + sin\omega t_1 cos\omega t_2 E\left[BA\right] + sin\omega t_1 sin\omega t_2 E\left[B^2\right]$$
$$= \sigma^2\left[cos\omega t_1 cos\omega t_2 + sin\omega t_1 sin\omega t_2\right]$$
$$= \sigma^2 cos\omega\left(t_1 - t_2\right) \qquad [\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \ \& \ E\left(AB\right) = E\left(BA\right) = 0]$$

$R_{XX}\left(t_1, t_2\right) =$ is a function of time difference.

$$E\left[Y\left(t\right)\right] = E\left[Bcos\omega t - Asin\omega t\right]$$
$$= E\left(B\right)cos\omega t - E\left(A\right)sin\omega t = 0$$
$$R_{YY}\left(t_1, t_2\right) = E\left[\left(B\,cos\omega t_1 - Asin\omega t_1\right)\left(B\,cos\omega t_2 - Asin\omega t_2\right)\right]$$
$$= E\left[B^2 cos\omega t_1 cos\omega t_2 - BA\,cos\omega t_1 sin\omega t_2 - AB\,sin\omega t_1 cos\omega t_2 + A^2 sin\omega t_1 sin\omega t_2\right]$$
$$= E\left(B^2\right)cos\omega t_1 cos\omega t_2 - E\left(BA\right)cos\omega t_1 sin\omega t_2 - E\left(AB\right)sin\omega t_1 cos\omega t_2 + E\left(A^2\right)sin\omega t_1 sin\omega t_2$$
$$= \sigma^2 cos\omega\left(t_1 - t_2\right) \qquad [\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \ \& \ E\left(AB\right) = E\left(BA\right) = 0]$$

$R_{YY}\left(t_1, t_2\right) =$ is a function of time difference.

$$R_{XY}\left(t_1, t_2\right) = E\left[X\left(t_1\right)Y\left(t_2\right)\right]$$
$$= E\left[\left(Acos\omega t_1 + Bsin\omega t_1\right)\left(Bcos\omega t_2 + Asin\omega t_2\right)\right]$$
$$= E\left[ABcos\omega t_1 cos\omega t_2 - A^2 cos\omega t_1 sin\omega t_2 + B^2 sin\omega t_1 cos\omega t_2 - BAsin\omega t_1 sin\omega t_2\right]$$
$$= \sigma^2\left[sin\omega t_1 cos\omega t_2 - cos\omega t_1 sin\omega t_2\right]$$
$$= \sigma^2 sin\omega\left(t_1 - t_2\right) \qquad [\because E\left(A^2\right) = E\left(B^2\right) = \sigma^2 \ \& \ E\left(AB\right) = E\left(BA\right) = 0]$$

$R_{XY}\left(t_1, t_2\right) =$ is a function of time difference.

Since $\left\{X\left(t\right)\right\}$ & $\left\{Y\left(t\right)\right\}$ are individually WSS & also $R_{XY}\left(t_1, t_2\right)$ is a function of time difference.

$\therefore$ The two random process $\left\{X\left(t\right)\right\}$ & $\left\{Y\left(t\right)\right\}$ are jointly WSS.

b). Write a note on Binomial process.

Solution:

Binomial Process can be defined as a sequence of partial sums $\left\{S_n / n = 1, 2, ...\right\}$ Where $S_n = X_1 + X_2 + ... + X_n$ Where X_i denotes 1 if the trial is success or 0 if the trial is failure.

As an example for a sample function of the binomial random process with $\left(x_1, x_2, ...\right) = \left(1, 1, 0, 0, 1, 0, 1, ...\right)$ is $\left(s_1, s_2, s_3, ...\right) = \left(1, 2, 2, 2, 3, 3, 4, ...\right)$. The process increments by 1 only at the discrete times t_i $t_i = iT$, $i = 1, 2, ...$

Properties

(i). Binomial process is Markovian

(ii). S_n is a binomial random variable so, $P(S_n = m) = nC_m p^m q^{n-m}$, $E[S_n] = np$ & $var[S_n] = np(1-p)$

(iii) The distribution of the number of slots m_i between i^{th} and $(i-1)^{th}$ arrival is geometric with parameter p starts from 0. The random variables m_i, $i = 1,2,...$ are mutually independent.

The geometric distribution is given by $p(1-p)^{i-1}, i = 1,2,...$

(iv) The binomial distribution of the process approaches poisson when n is large and p is small.

Problem 22. a). Describe Poisson process & show that the Poisson process is Markovian.

Solution:

If $\{X(t)\}$ represents the number of occurrences of a certain event in $(0,t)$ then the discrete random process $\{X(t)\}$ is called the Poisson process, provided the following postulates are satisfied

(i) $P[1 \; occumence \; in \; (t, t+\Delta t)] = \lambda \Delta t + o(\Delta t)$

(ii) $P[no \; occurrence \; in \; (t, t+\Delta t)] = 1 - \lambda \Delta t + o(\Delta t)$

(iii) $P[2 \; or \; more \; occurrences \; in \; (t, t+\Delta t)] = o(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0,t)$.

(v) The probability that the event occurs in a specified number of times $(t_0, t_0 + t)$ depends only on t, but not on t_0.

Consider

$$P\left[X\left(t_3 = n_3 / X(t_2) = n_2, X(t_1) = n_1\right)\right] = \frac{P\left[X(t_1) = n_1, X(t_2) = n_2, X(t_3) = n_3\right]}{P\left[X(t_1) = n_1, X(t_2) = n_2\right]}$$

$$= \frac{\dfrac{e^{-\lambda t_3} \lambda^{n_3} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1} (t_3 - t_2)^{n_3 - n_2}}{n_1!(n_2 - n_1)!(n_3 - n_2)!}}{\dfrac{e^{-\lambda t_2} \lambda^{n_2} t_1^{n_1} (t_2 - t_1)^{n_2 - n_1}}{n_1!(n_2 - n_1)!}}$$

$$= \frac{e^{-\lambda(t_3 - t_2)} \lambda^{n_3 - n_2} (t_3 - t_2)^{n_3 - n_2}}{(n_3 - n_2)!}$$

$$P\left[X\left(t_3 = n_3 / X(t_2) = n_2, X(t_1) = n_1\right)\right] = P\left[X(t_3) = n_3 / X(t_2) = n_2\right]$$

This means that the conditional probability distribution of $X(t_3)$ given all the past values $X(t_1) = n_1, X(t_2) = n_2$ depends only on the most recent values $X(t_2) = n_2$.

i.e., The Poisson process possesses Markov property.

b). State and establish the properties of Poisson process.

Solution:

(i). Sum of two independent poisson process is a poisson process.

Proof:

The moment generating function of the Poisson process is

$$M_{X(t)}(u) = E\left[e^{uX(t)}\right]$$
$$= \sum_{x=0}^{\infty} e^{ux} P\left[X(t) = x\right]$$
$$= \sum_{x=0}^{\infty} e^{ux} \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$
$$= e^{-\lambda t}\left[1 + \frac{e^u(\lambda t)^1}{1!} + \frac{e^{2u}(\lambda t)^2}{2!} + ...\right]$$
$$= e^{-\lambda t} e^{\lambda t e^u}$$
$$M_{X(t)}(u) = e^{\lambda t\left(e^u - 1\right)}$$

Let $X_1(t)$ and $X_2(t)$ be two independent Poisson processes

$\therefore$Their moment generating functions are,

$M_{X_1(t)}(u) = e^{\lambda_1 t\left(e^u - 1\right)}$ and $M_{X_2(t)}(u) = e^{\lambda_2 t\left(e^u - 1\right)}$

$$\therefore M_{X_1(t)+X_2(t)}(u) = M_{X_1(t)}(u) M_{X_2(t)}(u)$$
$$= e^{\lambda_1 t\left(e^u - 1\right)} e^{\lambda_2 t\left(e^u - 1\right)}$$
$$= e^{(\lambda_1 + \lambda_2)t\left(e^u - 1\right)}$$

$\therefore$By uniqueness of moment generating function, the process $\{X_1(t) + X_2(t)\}$ is a Poisson process with occurrence rate $(\lambda_1 + \lambda_2)$ per unit time.

(ii). Difference of two independent poisson process is not a poisson process.

Proof:

Let $X(t) = X_1(t) - X_2(t)$

$$E\{X(t)\} = E\{X_1(t)\} - E\{X_2(t)\}$$
$$= (\lambda_1 - \lambda_2)t$$
$$E\{X^2(t)\} = E\{X_1^2(t)\} + E\{X_2^2(t)\} - 2E\{X_1(t)\}E\{X_2(t)\}$$
$$= \left(\lambda_1^2 t^2 + \lambda_1 t\right) + \left(\lambda_2^2 t^2 + \lambda_2 t\right) - 2(\lambda_1 t)(\lambda_2 t)$$
$$= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$
$$\neq (\lambda_1 - \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

Recall that $E\{X^2(t)\}$ for a poisson process $\{X(t)\}$ with parameter λ is given by

$E\left\{X^2(t)\right\} = \lambda t + \lambda^2 t^2$

$\therefore X_1(t) - X_2(t)$ is not a poisson process.

(iii). The inter arrival time of a poisson process i.e., with the interval between two successive occurrences of a poisson process with parameter λ has an exponential distribution with mean $\frac{1}{\lambda}$.

Proof:

Let two consecutive occurrences of the event be E_i & E_{i+1}.

Let E_i take place at time instant t_i and T be the interval between the occurrences of E_i E_{i+1}.

Thus T is a continuous random variable.

$$\begin{aligned}
P(T > t) &= P\left\{Interval\,between\,occurrence\,of\; E_i \;and\; E_{i+1}\; exceeds\, t\right\} \\
&= P\left\{E_{i+1}\; does\,not\,occur\,upto\,the\,instant\left(t_i + t\right)\right\} \\
&= P\left\{No\,event\,occurs\,in\,the\,interval\left(t_i, t_i + t\right)\right\} \\
&= P\left\{X(t) = 0\right\} = P_0(t) \\
&= e^{-\lambda t}
\end{aligned}$$

$\therefore$ The cumulative distribution function of T is given by

$$F(t) = P\{T \le t\} = 1 - e^{-\lambda t}$$

$\therefore$ The probability density function is given by

$$f(t) = \lambda e^{-\lambda t}, \; (t \ge 0)$$

Which is an exponential distribution with mean $\frac{1}{\lambda}$.

Problem 23. a). If the process $\left\{N(t) : t \ge 0\right\}$ is a Poisson process with parameter λ, obtain $P\left[N(t) = n\right]$ and $E\left[N(t)\right]$.

Solution:

Let λ be the number of occurrences of the event in unit time.

Let $P_n(t)$ represent the probability of n occurrences of the event in the interval $(0, t)$.

i.e., $P_n(t) = P\left\{X(t) = n\right\}$

$$\begin{aligned}
\therefore P_n(t + \Delta t) &= P\left\{X(t + \Delta t) = n\right\} \\
&= P\left\{n \text{ occurences in the time } (0, t + \Delta t)\right\} \\
&= P\begin{cases} n \text{ occurences in the interval } (0,t) \text{ and no occurences in } (t, t+\Delta t) \text{ or} \\ n-1 \text{ occurences in the interval } (0,t) \text{ and } 1 \text{ occurences in } (t, t+\Delta t) \text{ or} \\ n-2 \text{ occurences in the interval } (0,t) \text{ and } 2 \text{ occurences in } (t, t+\Delta t) \text{ or...} \end{cases} \\
&= P_n(t)(1 - \lambda \Delta t) + P_{n-1}(t)\lambda \Delta t + 0 + \ldots
\end{aligned}$$

$$\therefore \frac{P_n(t+\Delta t)-P_n(t)}{\Delta t}=\lambda\left\{P_{n-1}(t)-P_n(t)\right\}$$

Taking the limits as $\Delta t \to 0$

$$\frac{d}{dt}P_n(t)=\lambda\left\{P_{n-1}(t)-P_n(t)\right\} \text{-------- (1)}$$

This is a linear differential equation.

$$\therefore P_n(t)e^{\lambda t}=\int_0^t \lambda P_{n-1}(t)e^{\lambda t} \text{------------- (2)}$$

Now taking $n=1$ we get

$$e^{\lambda t}P_1(t)=\lambda\int_0^t P_0(t)e^{\lambda t}dt \text{---------------- (3)}$$

Now, we have,

$$P_0(t+\Delta t)=P\left[0 \; occurences \; in \; (0,t+\Delta t)\right]$$

$$=P\left[0 \; occurences \; in \; (0,t) \; and \; 0 \; occurences \; in \; (t,t+\Delta t)\right]$$

$$=P_0(t)[1-\lambda t]$$

$$P_0(t+\Delta t)-P_0(t)=-P_0(t)(\lambda\Delta t)$$

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t}=-\lambda P_0(t)$$

$\therefore$ Taking limit $\Delta t \to 0$

$$\underset{\Delta t \to 0}{Lt}\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t}=-\lambda P_0(t)$$

$$\frac{dP_0(t)}{dt}=-\lambda P_0(t)$$

$$\frac{dP_0(t)}{P_0(t)}=-\lambda dt$$

$$\log P_0(t)=-\lambda t+c$$

$$P_0(t)=e^{-\lambda t+c}$$

$$P_0(t)=e^{-\lambda t}e^{c}$$

$$P_0(t)=e^{-\lambda t}A \text{----------------- (4)}$$

Putting $t=0$ we get

$$P_0(0)=e^0A=A$$

i.e., $A=1$

$\therefore$(4) we have

$$P_0(t)=e^{-\lambda t}$$

$\therefore$substituting in (3) we get

$$e^{\lambda t}P_1(t)=\lambda\int_0^t e^{-\lambda t}e^{\lambda t}dt$$

$$= \lambda \int_0^t dt = \lambda t$$

$P_1(t) = e^{-\lambda t} \lambda t$

Similarly $n = 2$ in (2) we have,

$$P_2(t) e^{\lambda t} = \lambda \int_0^t P_1(t) e^{\lambda t} dt$$

$$= \lambda \int_0^t e^{-\lambda t} \lambda t e^{\lambda t} dt$$

$$= \lambda^2 \left(\frac{t^2}{2} \right)$$

$$P_2(t) e^{\lambda t} = \frac{e^{-\lambda t} (\lambda t)^2}{2!}$$

Proceeding similarly we have in general

$$P_n(t) = P\{X(t) = n\} = \frac{e^{-\lambda t} (\lambda t)^n}{n!},\ n = 0, 1, ...$$

Thus the probability distribution of $X(t)$ is the Poisson distribution with parameter λt.

$$E[X(t)] = \lambda t.$$

.b). Find the mean and autocorrelation and auto covariance of the Poisson process.

Solution:

The probability law of the poisson process $\{X(t)\}$ is the same as that of a poisson distribution with parameter λt

$$E[X(t)] = \sum_{n=0}^{\infty} nP(X(t) = n)$$

$$= \sum_{n=0}^{\infty} n \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

$$= e^{-\lambda t} \sum_{n=1}^{\infty} \frac{\lambda t (\lambda t)^{n-1}}{(n-1)!}$$

$$= \lambda t e^{-\lambda t} e^{\lambda t}$$

$$E[X(t)] = \lambda t$$

$$Var[X(t)] = E[X^2(t)] - E[X(t)]^2$$

$$E\{X^2(t)\} = \sum_{n=0}^{\infty} n^2 P(X(t) = n)$$

$$= \sum_{n=0}^{\infty} (n^2 - n + n) \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1)\frac{e^{-\lambda t}(\lambda t)^n}{n!} + \sum_{n=0}^{\infty} n\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1)\frac{e^{-\lambda t}(\lambda t)^n}{n!} + \lambda t$$

$$= e^{-\lambda t}\left[\frac{(\lambda t)^2}{1} + \frac{(\lambda t)^3}{1!} + \frac{(\lambda t)^4}{2!} + \ldots\right] + \lambda t$$

$$= (\lambda t)^2 e^{-\lambda t} e^{\lambda t} + \lambda t$$

$$= (\lambda t)^2 + \lambda t$$

$$E\{X^2(t)\} = \lambda t + \lambda^2 t^2$$

$$\therefore Var[X(t)] = \lambda t$$

$$R_{XX}(t_1,t_2) = E\{X(t_1)X(t_2)\}$$

$$= E\{X(t_1)[X(t_2) - X(t_1) + X(t_1)]\}$$

$$= E\{X(t_1)[X(t_2) - X(t_1)] + E[X^2(t_1)]\}$$

$$= E[X(t_1)]E[X(t_2) - X(t_1)] + E[X^2(t_1)]$$

Since $\{X(t)\}$ is a process of independent increments.

$$= \lambda t_1[\lambda(t_2 - t_1)] + \lambda t_1 + \lambda t_1^2 \quad if \;\; t_2 \ge t_1 \quad (by-(1))$$

$$= \lambda^2 t_1 t_2 + \lambda t_1 \quad if \;\; t_2 \ge t_1$$

$$R_{XX}(t_1,t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1,t_2)$$

Auto Covariance

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - E\{X(t_1)\}E\{X(t_2)\}$$

$$= \lambda^2 t_1, t_2 + \lambda t_1 - \lambda^2 t_1 t_2$$

$$= \lambda t_1, \quad if \;\; t_2 \ge t_1$$

$$= \lambda \min(t_1,t_2)$$

Problem 24. a). Prove that the random process $X(t) = A\cos(\omega t + \theta)$. Where A, ω are constants θ is uniformly distributed random variable in $(0, 2\pi)$ is ergodic.

Solution:

Since θ is uniformly distribution in $(0, 2\pi)$.

$$f(\theta) = \frac{1}{2\pi}, \; 0 < \theta < 2\pi$$

Ensemble average same,

$$E[X(t)] = \int_0^{2\pi} \frac{1}{2\pi} cos(\omega t + \theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(cos\omega t\, cos\theta - sin\omega t\, sin\theta\right)d\theta$$
$$= \frac{1}{2\pi}\left[cos\omega t\, sin\theta + sin\omega t\, cos\theta\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left[sin\omega t - sin\omega t\right] = 0$$
$$E\left[X(t)\right] = 0$$
$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$
$$= E\left[cos(\omega t_1+\theta)cos(\omega t_2+\theta)\right]$$
$$= \frac{1}{2}E\left[cos(\omega t_1+\omega t_2+2\theta)+cos(\omega t_1-\omega t_2)\right]$$
$$= \frac{1}{2}\int_0^{2\pi}\frac{1}{2\pi}\left[cos(\omega t_1+\omega t_2+2\theta)+cos(\omega t_1-\omega t_2)\right]d\theta$$
$$= \frac{1}{4\pi}\left[\frac{sin(\omega t_1+\omega t_2+2\theta)+\theta cos(\omega(t_1-t_2))}{2}\right]_0^{2\pi}$$
$$= \frac{2\pi}{4\pi}cos(\omega(t_1-t_2))$$
$$R_{XX}(t_1,t_2) = \frac{1}{2}cos(\omega(t_1-t_2))$$

The time average can be determined by

$$\overline{X}(t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}cos(\omega t+\theta)\,dt$$
$$= \lim_{T\to\infty}\frac{1}{2T}\left[\frac{sin(\omega t+\theta)}{\omega}\right]_{-T}^{T}$$
$$= \lim_{T\to\infty}\frac{1}{2T\omega}\left[sin(\omega T+\theta)-sin(-\omega T+\theta)\right]$$
$$\overline{X}(t) = 0 \quad \left[As\ \ T->\infty\right]$$

The time auto correlation function of the process,

$$\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}X(t)X(t+\tau)\,dt$$
$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}cos(\omega t+\theta)cos(\omega t+\omega\tau+\theta)\,dt$$
$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{cos(\omega t+\omega\tau+2\theta)+cos\omega\tau}{2}\right]dt$$

$$= \lim_{T \to \infty} \frac{1}{4T}\left[\frac{sin\left(2\omega t+\omega t+2\theta\right)}{2}+tcos\omega t\right]_{-T}^{T}$$

$$= \lim_{T \to \infty} \frac{1}{4T}\left[\frac{sin\left(2\omega t+\omega t+2\theta\right)+sin\left(-2\omega T+\omega\tau+2\theta\right)}{2\omega}\right]+2Tcos\omega\tau$$

$$= \frac{cos\omega\tau}{2}$$

Since the ensemble average $=$ time average the given process is ergodic.

b). If the WSS process $\{X(t)\}$ is given by $X(t)=10\cos(100t+\theta)$ where θ is uniformly distributed over $(-\pi,\pi)$ prove that $\{X(t)\}$ is correlation ergodic.

Solution:

To Prove $\{X(t)\}$ is correlation ergodic it is enough to show that when $\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}X(t)X(t+\tau)\,dt = R_{XX}(\tau)$

$$R_{XX}(\tau)=E\left[X(t)X(t+\tau)\right]$$

$$=E\left[10cos\left(100t+\theta\right)10cos\left(100t+100\tau+\theta\right)\right]$$

$$=E\left\{100\left[\frac{cost\left(200t+100\tau+2\theta\right)+cos\left(100\tau\right)}{2}\right]\right\}$$

$$=50\int_{-\pi}^{\pi}\frac{1}{2\pi}\left[cos\left(200t+100\tau+2\theta\right)+cos\left(100\tau\right)\right]d\theta$$

$$=\frac{50}{2\pi}\left[\frac{sin\left(200t+100\tau+2\theta\right)}{2}+\theta cos\left(100\tau\right)\right]_{-\pi}^{\pi}$$

$$R_{XX}(\tau)=50cos(100\tau)$$

$$\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}X(t)X(t+\tau)\,dt$$

$$=\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}\left[10cos\left(100t+\theta\right)10cos\left(100t+100\tau+\theta\right)\right]dt$$

$$=\underset{T\to\infty}{Lt}\frac{25}{T}\int_{-T}^{T}\left[cos\left(200t+100\tau+2\theta\right)+cos\left(100\tau\right)\right]dt$$

$$=\underset{T\to\infty}{Lt}\frac{25}{T}\left[\frac{sin\left(200t+100\tau+2\theta\right)}{200}+t\,cos\left(100\tau\right)\right]_{-T}^{T}$$

$$\lim_{T\to\infty}\overline{X_T}=\lim_{T\to\infty}\frac{50}{T}Tcos\left(100\tau\right)$$

$$=50cos\left(100\tau\right)=\text{is a function of time difference}$$

$\therefore \{X(t)\}$ is correlation ergodic.

Problem 25. a). If the WSS process $\{X(t)\}$ is given by $X(t)=\cos(\omega t+\phi)$ where ϕ is uniformly distributed over $(-\pi,\pi)$ prove that $\{X(t)\}$ is correlation ergodic.

Solution:

To Prove $\{X(t)\}$ is correlation ergodic it is enough to show that when $\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}X(t)X(t+\tau)\,dt=R_{XX}(\tau)$

$$R_{XX}(\tau)=E\left[X(t)X(t+\tau)\right]$$

$$=E\left[cos(\omega t+\emptyset)cos(\omega t+\omega\tau+\emptyset)\right]$$

$$=E\left[\frac{cos(2\omega t+\omega\tau+2\emptyset)+c\,os(\omega\tau)}{2}\right]$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\left[cos(2\omega t+\omega\tau-2\emptyset)+cos\omega\tau\right]d\,\emptyset$$

$$=\frac{1}{4\pi}\left[\frac{sin(2\omega t+\omega\tau+2\emptyset)}{2}+\emptyset cos\omega\tau\right]_{-\pi}^{\pi}$$

$$R_{XX}(\tau)=\frac{1}{2}cos(\omega\tau)$$

Consider,

$$\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}X(t)X(t+\tau)\,dt$$

$$=\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}cos(\omega t+\emptyset)cos(\omega t+\omega\tau+\emptyset)\,dt$$

$$=\underset{T\to\infty}{Lt}\frac{1}{4T}\int_{-T}^{T}\left[cos(2\omega t+\omega\tau+2\emptyset)+cos(\omega\tau)\right]dt$$

$$=\underset{T\to\infty}{Lt}\frac{1}{4T}\left[\frac{sin(2\omega t+\omega\tau+2\emptyset)}{2\omega}+t\,cos(\omega\tau)\right]_{-T}^{T}$$

$$=\frac{1}{2}cos(\omega\tau)$$

$$\underset{T\to\infty}{\lim}\frac{1}{2\pi}\int_{-T}^{T}X(t)X(t+\tau)\,dt=\frac{1}{2}cos(\omega\tau)=R(\tau)$$

$\therefore \{X(t)\}$ is correlation ergodic.

b). A random process $\{X(t)\}$ defined as $\{X(t)\} = A\cos\omega t + B\sin\omega t$, where A & B are the random variables with $E(A) = E(B) = 0$ and $E(A^2) = E(B^2)$ & $E(AB) = 0$. Prove that the process is mean ergodic.

Solution:

To prove that the process is mean ergodic we have to shoe that the ensemble mean is same as the mean in the time sense.

Given $E(A) = E(B) = 0$-----------------1

$E(A^2) = E(B^2) = k(say)$ & $E(AB) = 0$-----2.

Ensemble mean is

$E[X(t)] = E[Acos\omega t + Bsin\omega t]$

$= E(A)cos\omega t + E(B)sin\omega t = 0$ Using (1)

Time Average

$$\overline{X}(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (A\,cos\omega t + B\,sin\omega t)\,dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\frac{Asin\omega t}{\omega} - \frac{Bcos\omega t}{\omega} \right]_{-T}^{T}$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\left(\frac{Asin\omega t}{\omega} - \frac{Bcos\omega t}{\omega} \right) - \left(\frac{Asin\omega(-T)}{\omega} - \frac{Bcos\omega(-T)}{\omega} \right) \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\frac{Asin\omega t}{\omega} - \frac{Bcos\omega t}{\omega} + \frac{Asin\omega T}{\omega} + \frac{Bcos\omega T}{\omega} \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\frac{2Asin\omega T}{\omega} \right]$$

$$= \frac{A}{\omega} \lim_{T \to \infty} \left[\frac{sin\omega T}{T} \right] = 0.$$

The ensemble mean =Time Average

Hence the process is mean ergodic.

26.a). Prove that in a Gaussian process if the variables are uncorrelated, then they are independent

Solution:

Let us consider the case of two variables X_{t_1} & X_{t_2}

If the are uncomelated then,

$r_{12} = r_{21} = r = 0$

Consequently the variance covariance matrix

$$\sum = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

Matrix of co-factors $=\sum_{ij}=\begin{pmatrix}\sigma_2^2 & 0\\ 0 & \sigma_1^2\end{pmatrix}$

$\therefore |\Sigma| = \sigma_1^2\sigma_2^2 - 0 = \sigma_1^2\sigma_2^2$

Now $(X-\mu)\Sigma^{-1}(X-\mu)^1 = (X-\mu)\frac{E_{ij}}{|\Sigma|}(X-\mu)^1$

Let us consider

$(X-\mu)E_{ij}(X-\mu)^1$

$$[X_1-\mu_1, X_2-\mu_2]\begin{bmatrix}\sigma_2^2 & 0\\ 0 & \sigma_1^2\end{bmatrix}\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix} = \left[(x_1-\mu_1)\sigma_2^2+0+0+(x_2-\mu_2)\sigma_1^2\right]\begin{bmatrix}x_1-\mu_1\\ x_2-\mu_2\end{bmatrix}$$

$$=(x_1-\mu_1)^2\sigma_2^2+(x_2-\mu_2)^2\sigma_1^2$$

Now the joint density of X_{t_1} & X_{t_2} is

$$f(x_1,x_2)=\frac{1}{(2\pi)^{2/2}\sqrt{\sigma_1^2\sigma_2^2}}\,e^{-\frac{1}{2\sigma_1^2\sigma_2^2}\left[(x_1-\mu_1)^2\sigma_1^2+(x_2-\mu_2)^2\sigma_2^2\right]}$$

$$=\frac{1}{(2\pi)\sigma_1\sigma_2}\,e^{\frac{-1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2+\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

$$=\frac{1}{2\pi\,\sigma_1\sigma_2}\,e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$$=\frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}\;\frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$f(x_1,x_2)=f(x_1)f(x_2)$

$\therefore X_{t_1}$ & X_{t_2} are independent.

b). If $\{X(t)\}$ is a Gaussian process with $\mu(t)=10$ & $C(t_1,t_2)=16\ e^{-|t_1-t_2|}$. Find the probability that (i) $X(10)\le 8$ and (ii) $|X(10)-X(6)|\le 4$.

Solution:

Since $\{X(t)\}$ is a Gaussion process, any member of the process is a normal random variable.

$\therefore X(10)$ is a normal RV with mean $\mu(10)=10$ and variance $C(10,10)=16$.

$$P\{X(10)\le 8\}=P\left\{\frac{X(10)-10}{4}\le -0.5\right\}$$

$=P\{Z\le -0.5\}$ (Where Z is the standard normal RV)

$=0.5-P\{Z\le 0.5\}$

$=0.5-0.1915$ (from normal tables)

$=0.3085$.

$X(10)-X(6)$ is also a normal R V With mean $\mu(10)-\mu(6)=10-10=0$

$$Var\{X(10)-X(6)\}=Var\{X(10)\}+Var\{X(6)\}-2Cov\{X(10),X(6)\}$$

$$=Cov(10,10)+Cov(6,6)-2Cov(10,6)$$

$$=16+16-2\times16e^{-4}=31.4139$$

$$P\{|X(10)-X(6)|\le4\}=P\left\{\frac{|X(10)-X(6)|}{5.6048}\le\frac{4}{5.6048}\right\}$$

$$=P\{|Z|\le0.7137\}$$

$$=2\times0.2611=0.5222.$$

Problem 27. a). Define a Markov chain. Explain how you would clarify the states and identify different classes of a Markov chain. Give example to each class.

Solution:

Markov Chain: If for all n,

$P\{X_n=a_n/X_{n-1}=a_{n-1},X_{n-2}=a_{n-2},...,X_0=a_0\}=P\{X_n=a_n/X_{n-1}=a_{n-1}\}$ then the process $\{X_n\}$, $n=0,1,2,...$ is called a Markov Chain.

<u>Classification of states of a Markov chain</u>

Irreducible: A Markov chain is said to be irreducible if every state can be reacted from every other state, where $P_{ij}^{(n)}>0$ for some n and for all $i\,\&\,j$.

Example: $\begin{bmatrix}0.3&0.7&0\\0.1&0.4&0.5\\0&0.2&0.8\end{bmatrix}$

Period: The Period d_i of a return state i is defined as the greatest common division of all m such that $P_{ij}^{(m)}>0$

i.e., $d_i=GCD\{m:p_{ij}^{(m)}>0\}$

State i is said to be periodic with period d_i if $d_i>1$ and a periodic if $d_i=1$.

Example:

$\begin{bmatrix}0&1\\1&0\end{bmatrix}$ So states are with period 2.

$\begin{bmatrix}\frac{1}{4}&\frac{3}{4}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}$ The states are aperiodic as period of each state is 1.

Ergodic: A non null persistent and aperiodic state is called ergodic.

Example:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}$$ Here all the states are ergodic.

b). The one-step T.P.M of a Markov chain $\{X_n;\, n = 0,1,2,...\}$ having state space $S = \{1,2,3\}$ is $P = \begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ and the initial distribution is $\pi_0 = (0.7,\ 0.2,\ 0.1)$.

Find (i) $P(X_2 = 3/X_0 = 1)$ (ii) $P(X_2 = 3)$ (iii) $P(X_3 = 2,\ X_2 = 3,\ X_1 = 3,\ X_0 = 1)$.

Solution:

$$(i)\, P(X_2 = 3/X_0 = 1) = P(X_2 = 3/X_1 = 3)P(X_1 = 3/X_0 = 1) + P(X_3 = 3/X_1 = 2)P(X_1 = 2/X_0 = 1)$$
$$+P(X_2 = 3/X_1 = 1)P(X_1 = 1/X_0 = 1)$$
$$= (0.3)(0.4) + (0.2)(0.5) + (0.4)(0.1) = 0.26$$

$$P^2 = P.P = \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

(ii). $P(X_2 = 3) = \sum_{i=1}^{3} P(X_2 = 3/X_0 = i)P(X_0 = i)$

$$= P(X_2 = 3/X_0 = 1)P(X_0 = 1) + P(X_2 = 3/X_0 = 2)P(X_0 = 2)$$
$$+ P(X_2 = 3/X_0 = 3)P(X_0 = 3)$$
$$= P_{13}^2 P(X_0 = 1) + P_{23}^2 P(X_0 = 2) + P_{33}^2 P(X_0 = 3)$$
$$= 0.26 \times 0.7 + 0.34 \times 0.2 + 0.29 \times 0.1 = 0.279$$

(iii). $P(X_3 = 2, X_2 = 3, X_1 = 3, X_0 = 1)$

$$= P[X_0 = 1, X_1 = 3, X_2 = 3]P[X_3 = 2/X_0 = 1, X_1 = 3, X_2 = 3]$$
$$= P[X_0 = 1, X_1 = 3, X_2 = 3]P[X_3 = 2/X_2 = 3]$$
$$= P[X_0 = 1, X_1 = 3]P[X_2 = 3/X_0 = 1, X_1 = 3]P[X_3 = 2/X_2 = 3]$$
$$= P[X_0 = 1, X_1 = 3]P[X_2 = 3/X_1 = 3]P[X_3 = 2/X_2 = 3]$$
$$= P[X_0 = 1]P[X_1 = 3/X_0 = 1]P[X_2 = 3/X_1 = 3]P[X_3 = 2/X_2 = 3]$$
$$= (0.4)(0.3)(0.4)(0.7) = 0.0336$$

Problem 28. a). Let $\{X_n;\ n=1,2,3,.....\}$ be a Markov chain with state space $S=\{0,1,2\}$ and 1 – step Transition probability matrix $P=\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}$ (i) Is the chain ergodic? Explain (ii) Find the invariant probabilities.

Solution:

$$P^2 = P.P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$P^3 = P^2P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \\ \frac{3}{16} & \frac{5}{8} & \frac{3}{16} \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} \end{pmatrix}$$

$P_{11}^{(3)} > 0,\ P_{13}^{(2)} > 0,\ P_{21}^{(2)} > 0,\ P_{22}^{(2)} > 0,\ P_{33}^{(2)} > 0$ and all other $P_{ij}^{(1)} > 0$

Therefore the chain is irreducible as the states are periodic with period 1

i.e., aperiodic since the chain is finite and irreducible, all are non null persistent

$\therefore$ The states are ergodic.

$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 \end{bmatrix}$$

$$\frac{\pi_1}{4} = \pi_0 \text{ -------------(1)}$$

$$\pi_0 + \frac{\pi_1}{2} + \pi_2 = \pi_1 \text{ -------------(2)}$$

$$\frac{\pi_1}{4} = \pi_2 \text{ -------------(3)}$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \text{ -------------(4)}$$

From (2) $\pi_0 + \pi_2 = \pi_1 - \frac{\pi_1}{2} = \frac{\pi_1}{2}$

$$\therefore \pi_0 + \pi_1 + \pi_2 = 1$$

$$\frac{\pi_1}{2} + \pi_1 = 1$$

$\frac{3\pi_1}{2} = 1$

$\pi_1 = \frac{2}{3}$

From (3) $\frac{\pi_1}{4} = \pi_2$

$\pi_2 = \frac{1}{6}$

Using (4) $\pi_0 + \frac{2}{3} + \frac{1}{6} = 1$

$\pi_0 + \frac{4+1}{6} = 1$

$\pi_0 + \frac{5}{6} = 1 \Rightarrow \pi_0 = \frac{1}{6}$

$\therefore \pi_0 + \frac{1}{6}, \pi_1 = \frac{2}{3}$ & $\pi_2 = \frac{1}{6}$.

b). Find the nature of the states of the Markov chain with the TPM $P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}$ and the state space $(1,2,3)$.

Solution:

$$P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^3 = P^2.P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} = P$$

$$P^4 = P^2.P^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = P^2$$

$\therefore P^{2n} = P^2$ & $P^{2n+1} = P$

Also $P_{00}^2 > 0,\ P_{01}^1 > 0,\ P_{02}^2 > 0$

$P_{10}^1 > 0,\ P_{11}^2 > 0,\ P_{12}^1 > 0$

$P_{20}^2 > 0,\ P_{21}^1 > 0,\ P_{22}^2 > 0$

$\Rightarrow$ The Markov chain is irreducible

Also $P_{ii}^2 = P_{ii}^4 = \ldots > 0$ for all i

$\Rightarrow$ The states of the chain have period 2. Since the chain is finite irreducible, all states are non null persistent. All states are not ergodic.

Problem 29. a). Three boys A, B and C are throwing a ball to each other. A always throws to B and B always throws to C, but C is as likely to throw the ball to B as to A. Find the TPM and classify the states.

Solution:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

$$P^2 = P \times P = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$P^3 = P^2 \times P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

For any $i = 2,3$

$P_{ii}^2 P_{ii}^3, \ldots > 0$

$\Rightarrow$ G.C.D of $2,3,5,\ldots = 1$

$\Rightarrow$ The period of 2 and 3 is 1. The state with period 1 is aperiodic all states are ergodic.

b). A man either drives a car or catches a train to go to office each day. He never goes 2 days in a row by train but he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that one the first day of the week, the man tossed a fair dice and drove to work iff a 6 appeared. Find the probability that he takes a train on the third day and also the probability that on the third day and also the probability that he drives to work in the long run.

Solution:

State Space = (train, car)

The TPM of the chain is

$$P = \begin{array}{c} \\ T \\ C \end{array} \begin{array}{c} \begin{array}{cc} T & C \end{array} \\ \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

P (traveling by car) = P (getting 6 in the toss of the die)= $\frac{1}{6}$

& P (traveling by train) = $\frac{5}{6}$

$$P^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$$

$$P^{(2)} = P^{(1)}P = \left(\frac{5}{6}, \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{1}{12}, \frac{11}{12}\right)$$

$$P^{(3)} = P^{(2)}P = \left(\frac{1}{12}, \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{11}{24}, \frac{13}{24}\right)$$

P (the man travels by train on the third day) = $\frac{11}{24}$

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution or stationary state distribution of the Markov chain.

By the property of π , $\pi P = \pi$

$$(\pi_1 \ \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1 \ \pi_2)$$

$$\frac{1}{2}\pi_2 = \pi_1$$

$$\pi_1 + \frac{1}{2}\pi_2 = \pi_2$$

& $\pi_1 + \pi_2 = 1$

Solving $\pi_1 = \frac{1}{3}$ & $\pi_2 = \frac{2}{3}$

P{The man travels by car in the long run }= $\frac{2}{3}$.

Problem 30 a). Three are 2 white marbles in urn A and 3 red marbles in urn B. At each step of the process, a marble is selected from each urn and the 2 marbles selected are inter changed. Let the state a_i of the system be the number of red marbles in A after i changes. What is the probability that there are 2 red marbles in A after 3 steps? In the long run, what is the probability that there are 2 red marbles in urn A?

Solution:

State Space $\{X_n\} = (0,1,2)$ Since the number of ball in the urn A is always 2.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

$X_n = 0$, $A = 2W$ (Marbles) $B = 3R$ (Marbles)

$X_{n+1} = 0 \qquad P_{00} = 0$

$X_{n+1} = 1 \qquad P_{01} = 1$

$X_{n+1} = 2 \qquad P_{02} = 0$

$X_n = 0$, $A = 1W \,\&\, 1R$ (Marbles) $B = 2R \,\&\, 1W$ (Marbles)

$X_{n+1} = 0 \qquad P_{10} = \frac{1}{6}$

$X_{n+1} = 1 \qquad P_{11} = \frac{1}{2}$

$X_{n+1} = 2 \qquad P_{12} = \frac{1}{3}$

$X_n = 2$, $A = 2R$ (Marbles) $B = 1R \,\&\, 2W$ (Marbles)

$X_{n+1} = 0 \qquad P_{20} = 0$

$X_{n+1} = 1 \qquad P_{21} = \frac{2}{3}$

$X_{n+1} = 2 \qquad P_{22} = \frac{1}{3}$

$P^{(0)} = (1,0,0)$ as there is not red marble in A in the beginning.

$P^{(1)} = P^{(0)}P = (0,\, 1,\, 0)$

$P^{(2)} = P^{(1)}P = \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$

$P^{(3)} = P^{(2)}P = \left(\frac{1}{12}, \frac{23}{36}, \frac{5}{18}\right)$

$\therefore$P (There are 2 red marbles in A after 3 steps) $= P\{X_3 = 2\} = P_2^{(3)} = \frac{5}{18}$

Let the stationary probability distribution of the chain be $\pi = (\pi_0,\, \pi_1,\, \pi_2)$.

By the property of π, $\pi P = \pi$ & $\pi_0 + \pi_1 + \pi_2 = 1$

$$\begin{pmatrix}\pi_0 & \pi_1 & \pi_2\end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix}\pi_0 & \pi_1 & \pi_2\end{pmatrix}$$

$\frac{1}{6}\pi_1 = \pi_0$

$\pi_0 + \frac{1}{2}\pi_1 + \frac{2}{3}\pi_2 = \pi_1$

$\frac{1}{3}\pi_1 + \frac{1}{3}\pi_2 = \pi_2$

& $\pi_0 + \pi_1 + \pi_2 = 1$

Solving $\pi_0 = \frac{1}{10}, \pi_1 = \frac{6}{10}, \pi_2 = \frac{3}{10}$

P {here are 2 red marbles in A in the long run} = 0.3.

b). A raining process is considered as a two state Markov chain. If it rains the state is 0 and if it does not rain the state is 1. The TPM is $P = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$. If the Initial distribution is $(0.4,\ 0.6)$. Find it chance that it will rain on third day assuming that it is raining today.

Solution:

$$P^2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}$$

$$= \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.32 \end{pmatrix}$$

P[rains on third day / it rains today $= P\left[X_3 = 0 / X_1 = 0\right]$

$$= P_{00}^2 = 0.44$$