

**MISRIMAL NAVAJEE MUNOTH JAIN ENGINEERING COLLEGE, CHENNAI**  
**DEPARTMENT OF MATHEMATICS**  
**PROBABILITY AND RANDOM PROCESSES (MA2261)**  
**SEMESTER –IV**  
**UNIT-II: TWO DIMENSIONAL RANDOM VARIABLES**  
**QUESTION BANK ANSWERS**

**Part.A**

**Problem 1.** Let  $X$  and  $Y$  have joint density function  $f(x, y) = 2$ ,  $0 < x < y < 1$ . Find the marginal density function. Find the conditional density function  $Y$  given  $X = x$ .

**Solution:**

Marginal density function of  $X$  is given by

$$\begin{aligned} f_X(x) &= f(x) = \int_{-\infty}^{\infty} f(x, y) dy \\ &= \int_x^1 f(x, y) dy = \int_x^1 2 dy = 2(y)_x^1 \\ &= 2(1-x), 0 < x < 1. \end{aligned}$$

Marginal density function of  $Y$  is given by

$$\begin{aligned} f_Y(y) &= f(y) = \int_{-\infty}^{\infty} f(x, y) dx \\ &= \int_0^y 2 dx = 2y, 0 < y < 1. \end{aligned}$$

Conditional distribution function of  $Y$  given  $X = x$  is  $f\left(\frac{y}{x}\right) = \frac{f(x, y)}{f(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}$ .

**Problem 2.** Verify that the following is a distribution function.  $F(x) = \begin{cases} 0 & , x < -a \\ \frac{1}{2} \left( \frac{x}{a} + 1 \right) & , -a < x < a \\ 1 & , x > a \end{cases}$ .

**Solution:**

$F(x)$  is a distribution function only if  $f(x)$  is a density function.

$$\begin{aligned} f(x) &= \frac{d}{dx}[F(x)] = \frac{1}{2a}, -a < x < a \\ \int_{-\infty}^{\infty} f(x) &= 1 \\ \therefore \int_{-a}^a \frac{1}{2a} dx &= \frac{1}{2a} [x]_{-a}^a = \frac{1}{2a} [a - (-a)] \end{aligned}$$

$$= \frac{1}{2a} \cdot 2a = 1.$$

Therefore, it is a distribution function.

**Problem 3.** Prove that  $\int_{x_1}^{x_2} f_X(x) dx = p(x_1 < x < x_2)$

**Solution:**

$$\begin{aligned}\int_{x_1}^{x_2} f_X(x) dx &= [F_X(x)]_{x_1}^{x_2} \\ &= F_X(x_2) - F_X(x_1) \\ &= P[X \leq x_2] - P[X \leq x_1] \\ &= P[x_1 \leq X \leq x_2]\end{aligned}$$

**Problem 4.** A continuous random variable  $X$  has a probability density function  $f(x) = 3x^2$ ,  $0 \leq x \leq 1$ . Find ' $a$ ' such that  $P(X \leq a) = P(X > a)$ .

**Solution:**

Since  $P(X \leq a) = P(X > a)$ , each must be equal to  $\frac{1}{2}$  because the probability is always 1.

$$\therefore P(X \leq a) = \frac{1}{2}$$

$$\Rightarrow \int_0^a f(x) dx = \frac{1}{2}$$

$$\int_0^a 3x^2 dx = \frac{1}{2} \Rightarrow 3 \left[ \frac{x^3}{3} \right]_0^a = a^3 = \frac{1}{2}.$$

$$\therefore a = \left( \frac{1}{2} \right)^{\frac{1}{3}}$$

**Problem 5.** Suppose that the joint density function

$$f(x, y) = \begin{cases} Ae^{-x-y}, & 0 \leq x \leq y, 0 \leq y \leq \infty \\ 0, & \text{otherwise} \end{cases} \quad \text{Determine } A.$$

**Solution:**

Since  $f(x, y)$  is a joint density function

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= 1. \\ \Rightarrow \int_0^{\infty} \int_0^y Ae^{-x-y} dx dy &= 1\end{aligned}$$

$$\Rightarrow A \int_0^{\infty} e^{-y} \left( \frac{e^{-x}}{-1} \right)^y dy = 1$$

$$\Rightarrow A \int_0^{\infty} [e^{-y} - e^{-2y}] dy = 1$$

$$\Rightarrow A \left[ \frac{e^{-y}}{-1} - \frac{e^{-2y}}{-2} \right]_0^{\infty} = 1$$

$$\Rightarrow A \left[ \frac{1}{2} \right] = 1 \Rightarrow A = 2$$

**Problem 6.** Examine whether the variables  $X$  and  $Y$  are independent, whose joint density function is  $f(x, y) = xe^{-x(y+1)}$ ,  $0 < x, y < \infty$ .

**Solution:**

The marginal probability function of  $X$  is

$$\begin{aligned} f_X(x) = f(x) &= \int_{-\infty}^{\infty} f(x, y) dy = \int_0^{\infty} xe^{-x(y+1)} dy \\ &= x \left[ \frac{e^{-x(y+1)}}{-x} \right]_0^{\infty} = -[0 - e^{-x}] = e^{-x}, \end{aligned}$$

The marginal probability function of  $Y$  is

$$\begin{aligned} f_Y(y) = f(y) &= \int_{-\infty}^{\infty} f(x, y) dx = \int_0^{\infty} xe^{-x(y+1)} dx \\ &= x \left\{ \left[ \frac{e^{-x(y+1)}}{-y-1} \right]_0^{\infty} - \left[ \frac{e^{-x(y+1)}}{(y+1)^2} \right]_0^{\infty} \right\} \\ &= \frac{1}{(y+1)^2} \end{aligned}$$

$$\text{Here } f(x).f(y) = e^{-x} \times \frac{1}{(y+1)^2} \neq f(x, y)$$

$\therefore X$  and  $Y$  are not independent.

**Problem 7.** If  $X$  has an exponential distribution with parameter 1. Find the pdf of  $y = \sqrt{x}$

**Solution:**

$$\text{Since } y = \sqrt{x}, x = y^2$$

Since  $X$  has an exponential distribution with parameter 1, the pdf of  $X$  is given by

$$f_X(x) = e^{-x}, x > 0 \quad [\because f(x) = \lambda e^{-\lambda x}, \lambda = 1]$$

$$\therefore f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$= e^{-x} 2y = 2ye^{-y^2}$$

$$f_Y(y) = 2ye^{-y^2}, \quad y > 0$$

**Problem 8.** If  $X$  is uniformly distributed random variable in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ , Find the probability density function of  $Y = \tan X$ .

**Solution:**

$$\text{Given } Y = \tan X \Rightarrow x = \tan^{-1} y$$

$$\therefore \frac{dx}{dy} = \frac{1}{1+y^2}$$

Since  $X$  is uniformly distribution in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ ,

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{\frac{\pi}{2} - \frac{\pi}{2}} = \frac{1}{\pi} \cdot \frac{1}{\frac{\pi}{2}}$$

$$f_X(x) = \frac{1}{\pi}, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

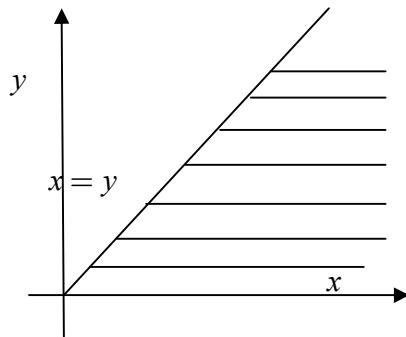
$$\text{Now } f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = \frac{1}{\pi} \left( \frac{1}{1+y^2} \right), \quad -\infty < y < \infty$$

$$\therefore f_Y(y) = \frac{1}{\pi(1+y^2)}, \quad -\infty < y < \infty$$

**Problem 9.** If the Joint probability density function of  $(x, y)$  is given by  $f(x, y) = 24y(1-x)$ ,  $0 \leq y \leq x \leq 1$  Find  $E(XY)$ .

**Solution:**

$$\begin{aligned} E(XY) &= \int_0^1 \int_y^1 xy f(x, y) dx dy \\ &= 24 \int_0^1 \int_y^1 xy^2 (1-x) dx dy \\ &= 24 \int_0^1 y^2 \left[ \frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3} \right] dy = \frac{4}{15}. \end{aligned}$$



**Problem 10.** If  $X$  and  $Y$  are random Variables, Prove that  $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

**Solution:**

$$\begin{aligned} \text{cov}(X, Y) &= E[(X - E(X))(Y - E(Y))] \\ &= E(XY - \bar{X}Y - \bar{Y}X + \bar{X}\bar{Y}) \\ &= E(XY) - \bar{X}E(Y) - \bar{Y}E(X) + \bar{X}\bar{Y} \\ &= E(XY) - \bar{X}\bar{Y} - \bar{X}\bar{Y} + \bar{X}\bar{Y} \end{aligned}$$

$$= E(XY) - E(X)E(Y) \quad [ \because E(X) = \bar{X}, E(Y) = \bar{Y} ]$$

**Problem 11.** If  $X$  and  $Y$  are independent random variables prove that  $\text{cov}(x, y) = 0$

**Solution:**

$$\text{cov}(x, y) = E(xy) - E(x)E(y)$$

But if  $X$  and  $Y$  are independent then  $E(xy) = E(x)E(y)$

$$\text{cov}(x, y) = E(x)E(y) - E(x)E(y)$$

$$\text{cov}(x, y) = 0.$$

**Problem 12.** Write any two properties of regression coefficients.

**Solution:**

1. Correlation coefficient is the geometric mean of regression coefficients

2. If one of the regression coefficients is greater than unity then the other should be less than 1.

$$b_{xy} = r \frac{\sigma_y}{\sigma_x} \text{ and } b_{yx} = r \frac{\sigma_x}{\sigma_y}$$

If  $b_{xy} > 1$  then  $b_{yx} < 1$ .

**Problem 13.** Write the angle between the regression lines.

**Solution:** The slopes of the regression lines are

$$m_1 = r \frac{\sigma_y}{\sigma_x}, m_2 = \frac{1}{r} \frac{\sigma_y}{\sigma_x}$$

If  $\theta$  is the angle between the lines, Then

$$\tan \theta = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2} \left[ \frac{1-r^2}{r} \right]$$

When  $r = 0$ , that is when there is no correlation between  $x$  and  $y$ ,  $\tan \theta = \infty$  (or)  $\theta = \frac{\pi}{2}$

and so the regression lines are perpendicular

When  $r = 1$  or  $r = -1$ , that is when there is a perfect correlation +ve or -ve,  $\theta = 0$  and so the lines coincide.

**Problem 14.** State central limit theorem

**Solution:**

If  $X_1, X_2, \dots, X_n$  is a sequence of independent random variable  $E(X_i) = \mu_i$  and  $\text{Var}(X_i) = \sigma_i^2, i = 1, 2, \dots, n$  and if  $S_n = X_1 + X_2 + \dots + X_n$  then under several conditions  $S_n$  follows a normal distribution with mean  $\mu = \sum_{i=1}^n \mu_i$  and variance  $\sigma^2 = \sum_{i=1}^n \sigma_i^2$  as  $n \rightarrow \infty$ .

**Problem 15.** i). Two random variables are said to be orthogonal if correlation is zero.

ii). If  $X = Y$  then correlation coefficient between them is 1.

### Part-B

**Problem 16.** a). The joint probability density function of a bivariate random variable  $(X, Y)$  is

$$f_{XY}(x, y) = \begin{cases} k(x+y), & 0 < x < 2, 0 < y < 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{where 'k' is a constant.}$$

- i. Find  $k$ .
- ii. Find the marginal density function of  $X$  and  $Y$ .
- iii. Are  $X$  and  $Y$  independent?
- iv. Find  $f_{Y/X}(y/x)$  and  $f_{X/Y}(x/y)$ .

**Solution:**

(i). Given the joint probability density function of a bivariate random variable  $(X, Y)$  is

$$f_{XY}(x, y) = \begin{cases} K(x+y), & 0 < x < 2, 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{Here } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy &= 1 \Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(x+y) dx dy = 1 \\ \int_0^2 \int_0^2 K(x+y) dx dy &= 1 \Rightarrow K \int_0^2 \left[ \frac{x^2}{2} + xy \right]_0^2 dy = 1 \\ &\Rightarrow K \int_0^2 (2+2y) dy = 1 \\ &\Rightarrow K [2y + y^2]_0^2 = 1 \\ &\Rightarrow K [8 - 0] = 1 \\ &\Rightarrow K = \frac{1}{8} \end{aligned}$$

(ii). The marginal p.d.f of  $X$  is given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy = \frac{1}{8} \int_0^2 (x+y) dy \\ &= \frac{1}{8} \left[ xy + \frac{y^2}{2} \right]_0^2 = \frac{1+x}{4} \end{aligned}$$

$\therefore$  The marginal p.d.f of  $X$  is

$$f_X(x) = \begin{cases} \frac{x+1}{4}, & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

The marginal p.d.f of  $Y$  is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \frac{1}{8} \int_0^2 (x+y) dx$$

$$\begin{aligned}
 &= \frac{1}{8} \left[ \frac{x^2}{2} + yx \right]_0^2 \\
 &= \frac{1}{8} [2 + 2y] = \frac{y+1}{4}
 \end{aligned}$$

$\therefore$  The marginal p.d.f of  $Y$  is

$$f_Y(y) = \begin{cases} \frac{y+1}{4}, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

(iii). To check whether  $X$  and  $Y$  are independent or not.

$$f_X(x)f_Y(y) = \frac{(x+1)}{4} \frac{(y+1)}{4} \neq f_{XY}(x,y)$$

Hence  $X$  and  $Y$  are not independent.

(iv). Conditional p.d.f  $f_{Y/X}\left(\frac{y}{x}\right)$  is given by

$$f_{Y/X}\left(\frac{y}{x}\right) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{1}{8}(x+y)}{\frac{1}{4}(x+1)} = \frac{1}{2} \frac{(x+y)}{(x+1)}$$

$$f_{Y/X}\left(\frac{y}{x}\right) = \frac{1}{2} \left( \frac{x+y}{x+1} \right), \quad 0 < x < 2, \quad 0 < y < 2$$

$$(v) P\left(0 < y < \frac{1}{2} \middle| x = 1\right) = \int_0^2 f_{Y/X}\left(\frac{y}{x=1}\right) dy$$

$$= \frac{1}{2} \int_0^{\frac{1}{2}} \frac{1+y}{2} dy = \frac{5}{32}.$$

**Problem 17.a).** If  $X$  and  $Y$  are two random variables having joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0 < x < 2, \quad 2 < y < 4 \\ 0, & \text{otherwise} \end{cases} \quad \text{Find (i) } P(X < 1 \cap Y < 3)$$

$$\text{(ii) } P(X+Y < 3) \quad \text{(iii) } P(X < 1 \middle| Y < 3).$$

b). Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If  $X$  denotes the number of white balls drawn and  $Y$  denotes the number of red balls drawn find the joint probability distribution of  $(X,Y)$ .

**Solution:**

a).

$$P(X < 1 \cap Y < 3) = \int_{y=-\infty}^{y=3} \int_{x=-\infty}^{x=1} f(x,y) dx dy$$

$$\begin{aligned}
 &= \int_{y=2}^{y=3} \int_{x=0}^{x=1} \frac{1}{8} (6-x-y) dx dy \\
 &= \frac{1}{8} \int_2^3 \int_0^1 (6-x-y) dx dy \\
 &= \frac{1}{8} \int_2^3 \left[ 6x - \frac{x^2}{2} - xy \right]_0^1 dy \\
 &= \frac{1}{8} \int_2^3 \left[ \frac{11}{2} - y \right] dy = \frac{1}{8} \left[ \frac{11y}{2} - \frac{y^2}{2} \right]_2^3 \\
 P(X < 1 \cap Y < 3) &= \frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii). } P(X + Y < 3) &= \int_0^{1/3-x} \int_0^1 \frac{1}{8} (6-x-y) dy dx \\
 &= \frac{1}{8} \int_0^1 \left[ 6y - xy - \frac{y^2}{2} \right]_2^{3-x} dx \\
 &= \frac{1}{8} \int_0^1 \left[ 6(3-x) - x(3-x) - \frac{(3-x)^2}{2} - [12-2x-2] \right] dx \\
 &= \frac{1}{8} \int_0^1 \left[ 18 - 6x - 3x + x^2 - \frac{(9+x^2-6x)}{2} - (10-2x) \right] dx \\
 &= \frac{1}{8} \int_0^1 \left[ 18 - 9x + x^2 - \frac{9}{2} - \frac{x^2}{2} + \frac{6x}{2} - 10 + 2x \right] dx \\
 &= \frac{1}{8} \int_0^1 \left[ \frac{7}{2} - 4x + \frac{x^2}{2} \right] dx \\
 &= \frac{1}{8} \left[ \frac{7x}{2} - \frac{4x^2}{2} + \frac{x^3}{6} \right]_0^1 = \frac{1}{8} \left[ \frac{7}{2} - 2 + \frac{1}{6} \right] \\
 &= \frac{1}{8} \left[ \frac{21-12+1}{6} \right] = \frac{1}{8} \left( \frac{10}{6} \right) = \frac{5}{24}.
 \end{aligned}$$

$$\text{(iii). } P(X < 1 \mid Y < 3) = \frac{P(x < 1 \cap y < 3)}{P(y < 3)}$$

$$\begin{aligned}
 \text{The Marginal density function of Y is } f_Y(y) &= \int_0^2 f(x, y) dx \\
 &= \int_0^2 \frac{1}{8} (6-x-y) dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{8} \left[ 6x - \frac{x^2}{2} - yx \right]_0^2 \\
 &= \frac{1}{8} [12 - 2 - 2y] \\
 &= \frac{5-y}{4}, \quad 2 < y < 4.
 \end{aligned}$$

$$\begin{aligned}
 P(X < 1 / Y < 3) &= \frac{\int_{x=0}^{x=1} \int_{y=2}^{y=3} \frac{1}{8} (6-x-y) dx dy}{\int_{y=2}^{y=3} f_Y(y) dy} \\
 &= \frac{\frac{3}{8}}{\int_2^3 \left( \frac{5-y}{4} \right) dy} = \frac{\frac{3}{8}}{\frac{1}{4} \left[ 5y - \frac{y^2}{2} \right]_2^3} \\
 &= \frac{3}{8} \times \frac{8}{5} = \frac{3}{5}.
 \end{aligned}$$

b). Let  $X$  takes 0, 1, 2 and  $Y$  takes 0, 1, 2 and 3.

$$\begin{aligned}
 P(X = 0, Y = 0) &= P(\text{drawing 3 balls none of which is white or red}) \\
 &= P(\text{all the 3 balls drawn are black}) \\
 &= \frac{4C_3}{9C_3} = \frac{4 \times 3 \times 2 \times 1}{9 \times 8 \times 7} = \frac{1}{21}.
 \end{aligned}$$

$$\begin{aligned}
 P(X = 0, Y = 1) &= P(\text{drawing 1 red ball and 2 black balls}) \\
 &= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}
 \end{aligned}$$

$$\begin{aligned}
 P(X = 0, Y = 2) &= P(\text{drawing 2 red balls and 1 black ball}) \\
 &= \frac{3C_2 \times 4C_1}{9C_3} = \frac{3 \times 2 \times 4 \times 3}{9 \times 8 \times 7} = \frac{1}{7}.
 \end{aligned}$$

$$\begin{aligned}
 P(X = 0, Y = 3) &= P(\text{all the three balls drawn are red and no white ball}) \\
 &= \frac{3C_3}{9C_3} = \frac{1}{84}
 \end{aligned}$$

$$\begin{aligned}
 P(X = 1, Y = 0) &= P(\text{drawing 1 White and no red ball}) \\
 &= \frac{2C_1 \times 4C_2}{9C_3} = \frac{\frac{2 \times 4 \times 3}{1 \times 2}}{\frac{9 \times 8 \times 7}{1 \times 2 \times 3}}
 \end{aligned}$$

$$= \frac{12 \times 1 \times 2 \times 3}{9 \times 8 \times 7} = \frac{1}{7}.$$

$P(X=1, Y=1) = P(\text{drawing 1 White and 1 red ball})$

$$= \frac{2C_1 \times 3C_1}{9C_3} = \frac{\frac{2 \times 3}{9 \times 8 \times 7}}{\frac{1 \times 2 \times 3}{9 \times 8 \times 7}} = \frac{2}{7}$$

$P(X=1, Y=2) = P(\text{drawing 1 White and 2 red ball})$

$$= \frac{2C_1 \times 3C_2}{9C_3} = \frac{\frac{2 \times 3 \times 2}{9 \times 8 \times 7}}{\frac{1 \times 2 \times 3}{9 \times 8 \times 7}} = \frac{1}{14}$$

$P(X=1, Y=3) = 0$  (Since only three balls are drawn)

$P(X=2, Y=0) = P(\text{drawing 2 white balls and no red balls})$

$$= \frac{2C_2 \times 4C_1}{9C_3} = \frac{1}{21}$$

$P(X=2, Y=1) = P(\text{drawing 2 white balls and no red balls})$

$$= \frac{2C_2 \times 3C_1}{9C_3} = \frac{1}{28}$$

$P(X=2, Y=2) = 0$

$P(X=2, Y=3) = 0$

The joint probability distribution of  $(X, Y)$  may be represented as

$X \backslash Y$	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

**Problem 18.a).** Two fair dice are tossed simultaneously. Let  $X$  denotes the number on the first die and  $Y$  denotes the number on the second die. Find the following probabilities.

(i)  $P(X+Y=8)$ , (ii)  $P(X+Y \geq 8)$ , (iii)  $P(X=Y)$  and (iv)  $P(X+Y=6 \mid Y=4)$ .

b) The joint probability mass function of a bivariate discrete random variable  $(X, Y)$  is given by the table.

$Y \backslash X$	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Find

- i. The marginal probability mass function of  $X$  and  $Y$ .
- ii. The conditional distribution of  $X$  given  $Y = 1$ .
- iii.  $P(X + Y < 4)$

**Solution:**

a). Two fair dice are thrown simultaneously

$$S = \left\{ \begin{matrix} (1,1)(1,2)\dots(1,6) \\ (2,1)(2,2)\dots(2,6) \\ \dots \quad \dots \quad \dots \\ (6,1)(6,2)\dots(6,6) \end{matrix} \right\}, \quad n(S) = 36$$

Let  $X$  denotes the number on the first die and  $Y$  denotes the number on the second die.

Joint probability density function of  $(X, Y)$  is  $P(X = x, Y = y) = \frac{1}{36}$  for

$x = 1, 2, 3, 4, 5, 6$  and  $y = 1, 2, 3, 4, 5, 6$

$$\begin{aligned} \text{(i)} \quad X + Y &= \{ \text{the events that the no is equal to } 8 \} \\ &= \{(2,6), (3,5), (4,4), (5,3), (6,2)\} \end{aligned}$$

$$\begin{aligned} P(X + Y = 8) &= P(X = 2, Y = 6) + P(X = 3, Y = 5) + P(X = 4, Y = 4) \\ &\quad + P(X = 5, Y = 3) + P(X = 6, Y = 2) \\ &= \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{5}{36} \end{aligned}$$

$$\text{(ii)} \quad P(X + Y \geq 8)$$

$$X + Y = \left\{ \begin{matrix} (2,6) \\ (3,5), (3,6) \\ (4,4), (4,5), (4,6) \\ (5,3), (5,4), (5,5), (5,6) \\ (6,2), (6,3), (6,4), (6,5), (6,6) \end{matrix} \right\}$$

$$\begin{aligned} \therefore P(X + Y \geq 8) &= P(X + Y = 8) + P(X + Y = 9) + P(X + Y = 10) \\ &\quad + P(X + Y = 11) + P(X + Y = 12) \\ &= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{15}{36} = \frac{5}{12} \end{aligned}$$

$$\text{(iii)} \quad P(X = Y)$$

$$\begin{aligned} P(X = Y) &= P(X = 1, Y = 1) + P(X = 2, Y = 2) + \dots + P(X = 6, Y = 6) \\ &= \frac{1}{36} + \frac{1}{36} + \dots + \frac{1}{36} = \frac{6}{36} = \frac{1}{6} \end{aligned}$$

$$\text{(iv)} \quad P(X + Y = 6 \mid Y = 4) = \frac{P(X + Y = 6 \cap Y = 4)}{P(Y = 4)}$$

$$\text{Now } P(X+Y=6 \cap Y=4) = \frac{1}{36}$$

$$P(Y=4) = \frac{6}{36}$$

$$\therefore P(X+Y=6 \mid Y=4) = \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6}.$$

b). The joint probability mass function of  $(X, Y)$  is

$X \backslash Y$	1	2	3	Total
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
Total	0.3	0.4	0.3	1

From the definition of marginal probability function

$$P_X(x_i) = \sum_{y_j} P_{XY}(x_i, y_j)$$

When  $X = 1$ ,

$$\begin{aligned} P_X(x_1) &= P_{XY}(1,1) + P_{XY}(1,2) \\ &= 0.1 + 0.2 = 0.3 \end{aligned}$$

When  $X = 2$ ,

$$\begin{aligned} P_X(x_2) &= P_{XY}(2,1) + P_{XY}(2,2) \\ &= 0.2 + 0.3 = 0.4 \end{aligned}$$

When  $X = 3$ ,

$$\begin{aligned} P_X(x_3) &= P_{XY}(3,1) + P_{XY}(3,2) \\ &= 0.2 + 0.1 = 0.3 \end{aligned}$$

$\therefore$  The marginal probability mass function of  $X$  is

$$P_X(x) = \begin{cases} 0.3 & \text{when } x=1 \\ 0.4 & \text{when } x=2 \\ 0.3 & \text{when } x=3 \end{cases}$$

The marginal probability mass function of  $Y$  is given by  $P_Y(y_j) = \sum_{x_i} P_{XY}(x_i, y_j)$

$$\begin{aligned} \text{When } Y=1, P_Y(y=1) &= \sum_{x_i=1}^3 P_{XY}(x_i, 1) \\ &= P_{XY}(1,1) + P_{XY}(2,1) + P_{XY}(3,1) \\ &= 0.1 + 0.1 + 0.2 = 0.4 \end{aligned}$$

$$\begin{aligned} \text{When } Y=2, P_Y(y=2) &= \sum_{x_i=1}^3 P_{XY}(x_i, 2) \\ &= P_{XY}(1,2) + P_{XY}(2,2) + P_{XY}(3,2) \\ &= 0.2 + 0.3 + 0.1 = 0.6 \end{aligned}$$

$\therefore$  Marginal probability mass function of  $Y$  is

$$P_Y(y) = \begin{cases} 0.4 & \text{when } y=1 \\ 0.6 & \text{when } y=2 \end{cases}$$

(ii) The conditional distribution of  $X$  given  $Y=1$  is given by

$$P\left(X=x \middle| Y=1\right) = \frac{P(X=x \cap Y=1)}{P(Y=1)}$$

From the probability mass function of  $Y$ ,  $P(y=1)=P_y(1)=0.4$

$$\begin{aligned} \text{When } X=1, P\left(X=1 \middle| Y=1\right) &= \frac{P(X=1 \cap Y=1)}{P(Y=1)} \\ &= \frac{P_{XY}(1,1)}{P_Y(1)} = \frac{0.1}{0.4} = 0.25 \end{aligned}$$

$$\text{When } X=2, P\left(X=2 \middle| Y=1\right) = \frac{P_{XY}(2,1)}{P_Y(1)} = \frac{0.1}{0.4} = 0.25$$

$$\text{When } X=3, P\left(X=3 \middle| Y=1\right) = \frac{P_{XY}(3,1)}{P_Y(1)} = \frac{0.2}{0.4} = 0.5$$

$$\begin{aligned} \text{(iii). } P(X+Y<4) &= P\{(x,y) / x+y < 4 \text{ Where } x=1,2,3; y=1,2\} \\ &= P\{(1,1), (1,2), (2,1)\} \\ &= P_{XY}(1,1) + P_{XY}(1,2) + P_{XY}(2,1) \\ &= 0.1 + 0.1 + 0.2 = 0.4 \end{aligned}$$

**Problem 19.a).** If  $X$  and  $Y$  are two random variables having the joint density function  $f(x,y) = \frac{1}{27}(x+2y)$  where  $x$  and  $y$  can assume only integer values 0, 1 and 2, find the conditional distribution of  $Y$  for  $X=x$ .

b). The joint probability density function of  $(X,Y)$  is given by

$$f_{XY}(x,y) = \begin{cases} xy^2 + \frac{x^2}{8}, & 0 \leq x \leq 2, \quad 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{Find (i) } P(X>1), \quad \text{(ii) } P(X < Y) \quad \text{and}$$

$$\text{(iii) } P(X+Y \leq 1)$$

**Solution:**

a). Given  $X$  and  $Y$  are two random variables having the joint density function

$$f(x,y) = \frac{1}{27}(x+2y) \quad \text{---(1)}$$

Where  $x=0,1,2$  and  $y=0,1,2$

Then the joint probability distribution  $X$  and  $Y$  becomes as follows

$X \backslash Y$	0	1	2	$f_1(x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{3}{27}$
1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{15}{27}$

The marginal probability distribution of  $X$  is given by  $f_1(X) = \sum_j P(x, y)$  and is calculated in the above column of above table.

The conditional distribution of  $Y$  for  $X$  is given by  $f_1(Y=y/X=x) = \frac{f(x,y)}{f_1(x)}$  and is obtained in the following table.

$Y \backslash X$	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$
2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

$$P(Y=0/X=0) = \frac{P(X=0, Y=0)}{P(X=0)} = \frac{0}{\frac{6}{27}} = 0$$

$$P(Y=1/X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = \frac{\frac{2}{27}}{\frac{6}{27}} = \frac{1}{3}$$

$$P(Y=2/X=0) = \frac{P(X=0, Y=2)}{P(X=0)} = \frac{\frac{4}{27}}{\frac{6}{27}} = \frac{2}{3}$$

$$P(Y=0/X=1) = \frac{P(X=1, Y=0)}{P(X=1)} = \frac{\frac{1}{27}}{\frac{9}{27}} = \frac{1}{9}$$

$$P(Y=1/X=1) = \frac{P(X=1, Y=1)}{P(X=1)} = \frac{\frac{3}{27}}{\frac{9}{27}} = \frac{3}{9} = \frac{1}{3}$$

$$P(Y=2/X=1) = \frac{P(X=1, Y=2)}{P(X=1)} = \frac{\frac{5}{27}}{\frac{9}{27}} = \frac{5}{9}$$

$$P(Y=0/X=2) = \frac{P(X=2, Y=0)}{P(X=2)} = \frac{\frac{2}{27}}{\frac{12}{27}} = \frac{1}{6}$$

$$P(Y=1/X=2) = \frac{P(X=2, Y=1)}{P(X=2)} = \frac{\frac{4}{27}}{\frac{12}{27}} = \frac{1}{3}$$

$$P(Y=2/X=2) = \frac{P(X=2, Y=2)}{P(X=2)} = \frac{\frac{6}{27}}{\frac{12}{27}} = \frac{1}{2}$$

b). Given the joint probability density function of  $(X, Y)$  is  $f_{XY}(x+y) = xy^2 + \frac{x^2}{8}$ ,  $0 \leq x \leq 2$ ,  $0 \leq y \leq 1$

$$(i). P(X > 1) = \int_1^\infty f_X(x) dx$$

The Marginal density function of  $X$  is  $f_X(x) = \int_0^1 f(x, y) dy$

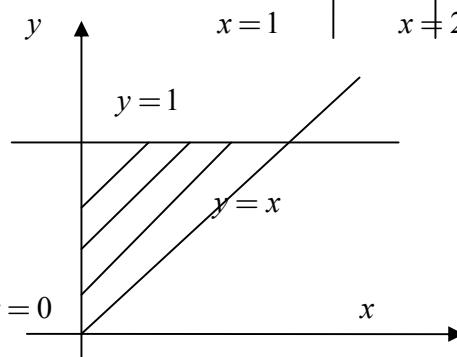
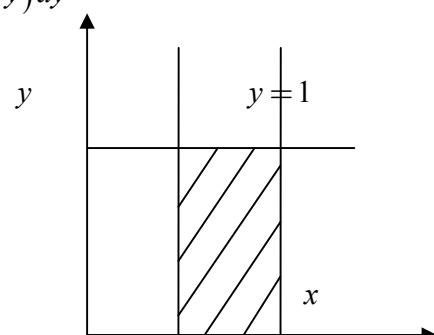
$$\begin{aligned} f_X(x) &= \int_0^1 \left( xy^2 + \frac{x^2}{8} \right) dy \\ &= \left[ \frac{xy^3}{3} + \frac{x^2 y}{8} \right]_0^1 = \frac{x}{3} + \frac{x^2}{8}, \quad 1 < x < 2 \end{aligned}$$

$$P(X > 1) = \int_1^2 \left( \frac{x}{3} + \frac{x^2}{8} \right) dx$$

$$= \left[ \frac{x^2}{6} + \frac{x^3}{24} \right]_1^2 = \frac{19}{24}.$$

$$(ii). P(X < Y) = \iint_{R_2} f_{XY}(x, y) dxdy$$

$$\begin{aligned} P(X < Y) &= \int_{y=0}^1 \int_{x=0}^y \left( xy^2 + \frac{x^2}{8} \right) dxdy \\ &= \int_0^1 \left[ \frac{x^2 y^2}{2} + \frac{x^3}{24} \right]_0^y dy \end{aligned}$$

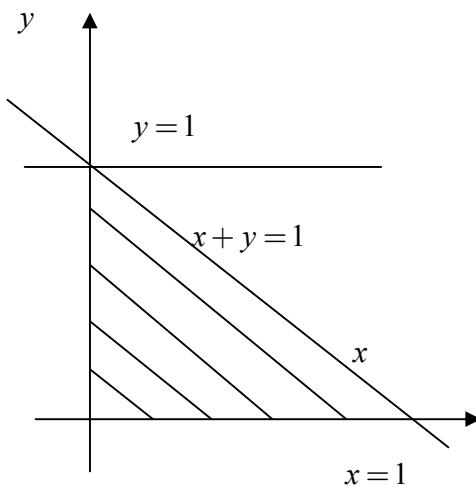


$$= \int_0^1 \left( \frac{y^4}{2} + \frac{y^3}{24} \right) dy = \left[ \frac{y^5}{10} + \frac{y^4}{96} \right]_0^1 \\ = \frac{1}{10} + \frac{1}{96} = \frac{96+10}{960} = \frac{53}{480}$$

$$(iii) P(X+Y \leq 1) = \iint_{R_3} f_{XY}(x,y) dx dy$$

Where  $R_3$  is the region

$$P(X+Y \leq 1) = \int_{y=0}^1 \int_{x=0}^{1-y} \left( xy^2 + \frac{x^2}{8} \right) dx dy \\ = \int_{y=0}^1 \left[ \left( \frac{x^2 y^2}{2} + \frac{x^3}{24} \right) \right]_0^{1-y} dy \\ = \int_{y=0}^1 \left( \frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24} \right) dy \\ = \int_0^1 \left( \frac{(1+y^2-2y)y^2}{2} + \frac{(1-y)^3}{24} \right) dy \\ = \left[ \left[ \frac{y^3}{3} + \frac{y^5}{5} - \frac{2y^2}{4} \right] \frac{1}{2} + \frac{(1-y)^4}{96} \right]_0^1 \\ = \frac{1}{6} + \frac{1}{10} - \frac{1}{4} + \frac{1}{96} = \frac{13}{480}.$$



**Problem20** a). If the joint distribution functions of  $X$  and  $Y$  is given by

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

- i. Find the marginal density of  $X$  and  $Y$ .
- ii. Are  $X$  and  $Y$  independent.
- iii.  $P(1 < X < 3, 1 < Y < 2)$ .

b). The joint probability distribution of  $X$  and  $Y$  is given by

$$f(x,y) = \begin{cases} \frac{6-x-y}{8}, & 0 < x < 2, 2 < y < 4 \\ 0, & \text{otherwise} \end{cases}. \text{ Find } P(1 < Y < 3 | X = 2).$$

**Solution:**

$$\text{a). Given } F(x,y) = (1-e^{-x})(1-e^{-y}) \\ = 1 - e^{-x} - e^{-y} + e^{-(x+y)}$$

The joint probability density function is given by

$$\begin{aligned}
 f(x,y) &= \frac{\partial^2 F(x,y)}{\partial x \partial y} \\
 &= \frac{\partial^2}{\partial x \partial y} \left[ 1 - e^{-x} - e^{-y} + e^{-(x+y)} \right] \\
 &= \frac{\partial}{\partial x} \left[ e^{-y} - e^{-(x+y)} \right] \\
 \therefore f(x,y) &= \begin{cases} e^{-(x+y)}, & x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

(ii) The marginal probability function of  $X$  is given by

$$\begin{aligned}
 f(x) &= f_X(x) \\
 &= \int_{-\infty}^{\infty} f(x,y) dy = \int_0^{\infty} e^{-(x+y)} dy \\
 &= \left[ \frac{e^{-(x+y)}}{-1} \right]_0^{\infty} \\
 &= \left[ -e^{-(x+y)} \right]_0^{\infty} \\
 &= e^{-x}, \quad x > 0
 \end{aligned}$$

The marginal probability function of  $Y$  is

$$\begin{aligned}
 f(y) &= f_Y(y) \\
 &= \int_{-\infty}^{\infty} f(x,y) dx \\
 &= \int_0^{\infty} e^{-(x+y)} dx = \left[ -e^{-(x+y)} \right]_0^{\infty} \\
 &= e^{-y}, \quad y > 0
 \end{aligned}$$

$$\therefore f(x)f(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x,y)$$

$\therefore X$  and  $Y$  are independent.

(iii)  $P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3) \times P(1 < Y < 2)$  [Since  $X$  and  $Y$  are independent]

$$\begin{aligned}
 &= \int_1^3 f(x) dx \times \int_1^2 f(y) dy \\
 &= \int_1^3 e^{-x} dx \times \int_1^2 e^{-y} dy \\
 &= \left[ \frac{e^{-x}}{-1} \right]_1^3 \times \left[ \frac{e^{-y}}{-1} \right]_1^2 \\
 &= (-e^{-3} + e^{-1})(-e^{-2} + e^{-1}) \\
 &= e^{-5} - e^{-4} - e^{-3} + e^{-2}
 \end{aligned}$$

$$b). P(1 < Y < 3 | X = 2) = \int_1^3 f(y/x=2) dy$$

$$\begin{aligned} f_x(x) &= \int_2^4 f(x,y) dy \\ &= \int_2^4 \left( \frac{6-x-y}{8} \right) dy \\ &= \frac{1}{8} \left( 6y - xy - \frac{y^2}{2} \right)_2^4 \\ &= \frac{1}{8} (16 - 4x - 10 + 2x) \end{aligned}$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\frac{6-x-y}{8}}{\frac{6-2x}{8}} = \frac{6-x-y}{6-2x}$$

$$\begin{aligned} P(1 < Y < 3 | X = 2) &= \int_1^3 f(y/x=2) dy \\ &= \int_2^3 \left( \frac{4-y}{2} \right) dy \\ &= \frac{1}{2} \left[ 4y - \frac{y^2}{2} \right]_2^3 \\ &= \frac{1}{2} \left[ 4y - \frac{y^2}{2} \right]_2^3 = \frac{1}{2} \left[ 14 - \frac{17}{2} \right] = \frac{11}{4}. \end{aligned}$$

**Problem 21.** a). Two random variables  $X$  and  $Y$  have the following joint probability density function  $f(x,y) = \begin{cases} 2-x-y, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$ . Find the marginal probability density function of  $X$  and  $Y$ . Also find the covariance between  $X$  and  $Y$ .

b). If  $f(x,y) = \frac{6-x-y}{8}, 0 \leq x \leq 2, 2 \leq y \leq 4$  for a bivariate  $(X,Y)$ , find the correlation coefficient

**Solution:**

a) Given the joint probability density function  $f(x,y) = \begin{cases} 2-x-y, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$

$$\begin{aligned} \text{Marginal density function of } X \text{ is } f_x(x) &= \int_{-\infty}^{\infty} f(x,y) dy \\ &= \int_0^1 (2-x-y) dy \end{aligned}$$

$$\begin{aligned}
 &= \left[ 2y - xy - \frac{y^2}{2} \right]_0^1 \\
 &= 2 - x - \frac{1}{2} \\
 f_x(x) &= \begin{cases} \frac{3}{2} - x, & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

Marginal density function of  $Y$  is  $f_Y(y) = \int_0^1 (2 - x - y) dx$

$$\begin{aligned}
 &= \left[ 2x - \frac{x^2}{2} - xy \right]_0^1 \\
 &= \frac{3}{2} - y \\
 f_Y(y) &= \begin{cases} \frac{3}{2} - y, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}$$

Covariance of  $(X, Y) = Cov(X, Y) = E(XY) - E(X)E(Y)$

$$\begin{aligned}
 E(X) &= \int_0^1 xf_X(x) dx = \int_0^1 x \left( \frac{3}{2} - x \right) dx = \left[ \frac{3}{2} \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{5}{12} \\
 E(Y) &= \int_0^1 yf_Y(y) dy = \int_0^1 y \left( \frac{3}{2} - y \right) dy = \frac{5}{12}
 \end{aligned}$$

$Cov(X, Y) = E(XY) - E(X)E(Y)$

$$\begin{aligned}
 E(XY) &= \int_0^1 \int_0^1 xy f(x, y) dxdy \\
 &= \int_0^1 \int_0^1 xy (2 - x - y) dxdy \\
 &= \int_0^1 \int_0^1 (2xy - x^2y - xy^2) dxdy \\
 &= \int_0^1 \left[ \frac{2x^2y}{2} - \frac{x^3}{3}y - \frac{x^2}{2}y^2 \right]_0^1 dy \\
 &= \int_0^1 \left( y - \frac{1}{3}y - \frac{y^2}{2} \right) dy \\
 &= \left[ \frac{y^2}{2} - \frac{y}{3} - \frac{y^3}{6} \right]_0^1 = \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned} \text{Cov}(X, Y) &= \frac{1}{6} - \frac{5}{12} \times \frac{5}{12} \\ &= \frac{1}{6} - \frac{25}{144} = -\frac{1}{144}. \end{aligned}$$

b). Correlation coefficient  $\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}$

Marginal density function of  $X$  is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_2^4 \left( \frac{6-x-y}{8} \right) dy = \frac{6-2x}{8}$$

Marginal density function of  $Y$  is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^2 \left( \frac{6-x-y}{8} \right) dx = \frac{10-2y}{8}$$

$$\begin{aligned} \text{Then } E(X) &= \int_0^2 x f_X(x) dx = \int_0^2 x \left( \frac{6-2x}{8} \right) dx \\ &= \frac{1}{8} \left[ \frac{6x^2}{2} - \frac{2x^3}{3} \right]_0^2 \\ &= \frac{1}{8} \left[ 12 - \frac{16}{3} \right] = \frac{1}{8} \times \frac{20}{3} = \frac{5}{6} \end{aligned}$$

$$E(Y) = \int_2^4 y \left( \frac{10-2y}{8} \right) dy = \frac{1}{8} \left[ \frac{10y^2}{2} - \frac{2y^3}{3} \right]_2^4 = \frac{17}{6}$$

$$E(X^2) = \int_0^2 x^2 f_X(x) dx = \int_0^2 x^2 \left( \frac{6-2x}{8} \right) dx = \frac{1}{8} \left[ \frac{6x^3}{3} - \frac{2x^4}{4} \right]_0^2 = 1$$

$$E(Y^2) = \int_2^4 y^2 \left( \frac{10-2y}{8} \right) dy = \frac{1}{8} \left[ \frac{10y^3}{3} - \frac{2y^4}{4} \right]_2^4 = \frac{25}{3}$$

$$\text{Var}(X) = \sigma_X^2 = E(X^2) - [E(X)]^2 = 1 - \left( \frac{5}{6} \right)^2 = \frac{11}{36}$$

$$\text{Var}(Y) = \sigma_Y^2 = E(Y^2) - [E(Y)]^2 = \frac{25}{3} - \left( \frac{17}{6} \right)^2 = \frac{11}{36}$$

$$\begin{aligned} E(XY) &= \int_2^4 \int_0^2 xy \left( \frac{6-x-y}{8} \right) dx dy \\ &= \frac{1}{8} \int_2^4 \left[ \frac{6x^2 y}{2} - \frac{x^3 y}{3} - \frac{x^2 y^2}{2} \right]_0^2 dy \\ &= \frac{1}{8} \int_2^4 \left( 12y - \frac{8}{3}y - 2y^2 \right) dy = \frac{1}{8} \left[ \frac{12y^2}{2} - \frac{8}{3} \frac{y^3}{2} - \frac{2y^3}{3} \right]_2^4 \end{aligned}$$

$$= \frac{1}{8} \left[ 96 - \frac{64}{3} - \frac{128}{3} - 24 + \frac{16}{3} + \frac{16}{3} \right] = \frac{1}{8} \left[ \frac{56}{3} \right]$$

$$E(XY) = \frac{7}{3}$$

$$\rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = \frac{\frac{7}{3} - \left(\frac{5}{6}\right)\left(\frac{17}{6}\right)}{\frac{\sqrt{11}}{6} \frac{\sqrt{11}}{6}}$$

$$\rho_{XY} = -\frac{1}{11}.$$

**Problem 22 a).** Let the random variables  $X$  and  $Y$  have pdf

$f(x, y) = \frac{1}{3}$ ,  $(x, y) = (0, 0), (1, 1), (2, 0)$ . Compute the correlation coefficient.

b) Let  $X_1$  and  $X_2$  be two independent random variables with means 5 and 10 and standard deviations 2 and 3 respectively. Obtain the correlation coefficient of  $UV$  where  $U = 3X_1 + 4X_2$  and  $V = 3X_1 - X_2$ .

**Solution:**

a). The probability distribution is

$\backslash$	$X$	0	1	2	$P(Y)$
$Y$					
0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	
1	0	$\frac{1}{3}$	0	$\frac{1}{3}$	
0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	
$P(X)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		

$$E(X) = \sum_i x_i p_i(x_i) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(2 \times \frac{1}{3}\right) = 1$$

$$E(Y) = \sum_j y_j p_j(y_j) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(0 \times \frac{1}{3}\right) = \frac{1}{3}$$

$$E(X^2) = \sum_i x_i^2 p(x_i) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{3}\right) = \frac{5}{3}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{5}{3} - 1 = \frac{2}{3}$$

$$E(Y^2) = \sum_j y_j^2 p(y_j) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{3}\right) + \left(0 \times \frac{1}{3}\right) = \frac{1}{3}$$

$$\therefore V(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

$$\text{Correlation coefficient } \rho_{XY} = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}}$$

$$E(XY) = \sum_i \sum_j x_i y_j p(x_i, y_j)$$

$$= 0.0 \cdot \frac{1}{3} + 0.1 \cdot 0 + 1.0 \cdot 0 + 1.1 \cdot \frac{1}{3} + 1.2 \cdot 0 + 0.0 \cdot 0 + 0.1 \cdot 0 + 0.2 \cdot \frac{1}{3} = \frac{1}{3}$$

$$\rho_{XY} = \frac{\frac{1}{3} - (1)\left(\frac{1}{3}\right)}{\sqrt{\frac{2}{3} \times \frac{2}{9}}} = 0$$

Correlation coefficient = 0.

b). Given  $E(X_1) = 5$ ,  $E(X_2) = 10$

$$V(X_1) = 4, V(X_2) = 9$$

Since  $X$  and  $Y$  are independent  $E(XY) = E(X)E(Y)$

$$\text{Correlation coefficient} = \frac{E(UV) - E(U)E(V)}{\sqrt{Var(U)Var(V)}}$$

$$E(U) = E(3X_1 + 4X_2) = 3E(X_1) + 4E(X_2)$$

$$= (3 \times 5) + (4 \times 10) = 15 + 40 = 55.$$

$$E(V) = E(3X_1 - X_2) = 3E(X_1) - E(X_2)$$

$$= (3 \times 5) - 10 = 15 - 10 = 5$$

$$E(UV) = E[(3X_1 + 4X_2)(3X_1 - X_2)]$$

$$= E[9X_1^2 - 3X_1X_2 + 12X_1X_2 - 4X_2^2]$$

$$= 9E(X_1^2) - 3E(X_1X_2) + 12E(X_1X_2) - 4E(X_2^2)$$

$$= 9E(X_1^2) + 9E(X_1X_2) - 4E(X_2^2)$$

$$= 9E(X_1^2) + 9E(X_1)E(X_2) - 4E(X_2^2)$$

$$= 9E(X_1^2) + 450 - 4E(X_2^2)$$

$$V(X_1) = E(X_1^2) - [E(X_1)]^2$$

$$E(X_1^2) = V(X_1) + [E(X_1)]^2 = 4 + 25 = 29$$

$$E(X_2^2) = V(X_2) + [E(X_2)]^2 = 9 + 100 = 109$$

$$\begin{aligned}\therefore E(UV) &= (9 \times 29) + 450 - (4 \times 109) \\ &= 261 + 450 - 436 = 275\end{aligned}$$

$$\begin{aligned}Cov(U,V) &= E(UV) - E(U)E(V) \\ &= 275 - (5 \times 55) = 0\end{aligned}$$

Since  $Cov(U,V) = 0$ , Correlation coefficient = 0.

**Problem 23.** a). Let the random variable  $X$  has the marginal density function  $f(x) = 1, -\frac{1}{2} < x < \frac{1}{2}$  and let the conditional density of  $Y$  be

$$f(y/x) = \begin{cases} 1, & x < y < x+1, -\frac{1}{2} < x < 0 \\ 1, & -x < y < 1-x, 0 < x < \frac{1}{2} \end{cases} \text{ Prove that the variables } X \text{ and } Y \text{ are uncorrelated.}$$

b). Given  $f(x,y) = xe^{-x(y+1)}$ ,  $x \geq 0, y \geq 0$ . Find the regression curve of  $Y$  on  $X$ .

**Solution:**

a). We have  $E(X) = \int_{-\frac{1}{2}}^{\frac{1}{2}} xf(x)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} xdx = \left[ \frac{x^2}{2} \right]_{-\frac{1}{2}}^{\frac{1}{2}} = 0$

$$\begin{aligned}E(XY) &= \int_{-\frac{1}{2}}^0 \int_x^{x+1} xy dx dy + \int_0^{\frac{1}{2}} \int_{-x}^{1-x} xy dx dy \\ &= \int_{-\frac{1}{2}}^0 x \left[ \int_x^{x+1} y dy \right] dx + \int_0^{\frac{1}{2}} x \left[ \int_{-x}^{1-x} y dy \right] dx \\ &= \frac{1}{2} \int_{-\frac{1}{2}}^0 x(2x+1) dx + \frac{1}{2} \int_0^{\frac{1}{2}} x(1-2x) dx \\ &= \frac{1}{2} \left[ \frac{2x^3}{3} + \frac{x^2}{2} \right]_{-\frac{1}{2}}^0 + \frac{1}{2} \left[ \frac{x^2}{2} - \frac{2x^3}{3} \right]_0^{\frac{1}{2}} = 0\end{aligned}$$

Since  $Cov(X,Y) = E(XY) - E(X)E(Y) = 0$ , the variables  $X$  and  $Y$  are uncorrelated.

b). Regression curve of  $Y$  on  $X$  is  $E(y/x)$

$$E(y/x) = \int_{-\infty}^{\infty} y f(y/x) dy$$

$$f(y/x) = \frac{f(x,y)}{f_X(x)}$$

$$\begin{aligned}
 \text{Marginal density function } f_X(x) &= \int_0^\infty f(x, y) dy \\
 &= x \int_0^\infty e^{-x(y+1)} dy \\
 &= x \left[ \frac{e^{-x(y+1)}}{-x} \right]_0^\infty = e^{-x}, \quad x \geq 0
 \end{aligned}$$

$$\text{Conditional pdf of } Y \text{ on } X \text{ is } f\left(\frac{y}{x}\right) = \frac{f(x, y)}{f_X(x)} = \frac{x e^{-xy-x}}{e^{-x}} = x e^{-xy}$$

The regression curve of  $Y$  on  $X$  is given by

$$\begin{aligned}
 E\left(\frac{y}{x}\right) &= \int_0^\infty y x e^{-xy} dy \\
 &= x \left[ y \frac{e^{-xy}}{-x} - \frac{e^{-xy}}{x^2} \right]_0^\infty
 \end{aligned}$$

$$E\left(\frac{y}{x}\right) = \frac{1}{x} \Rightarrow y = \frac{1}{x} \text{ and hence } xy = 1.$$

$$\text{Problem 24. a). Given } f(x, y) = \begin{cases} \frac{x+y}{3}, & 0 < x < 1, 0 < y < 2 \\ 0, & \text{otherwise} \end{cases},$$

obtain the regression of  $Y$  on  $X$  and  $X$  on  $Y$ .

b). Distinguish between correlation and regression Analysis

**Solution:**

a). Regression of  $Y$  on  $X$  is  $E\left(\frac{Y}{X}\right)$

$$\begin{aligned}
 E\left(\frac{Y}{X}\right) &= \int_{-\infty}^{\infty} y f\left(\frac{y}{x}\right) dy \\
 f\left(\frac{Y}{X}\right) &= \frac{f(x, y)}{f_X(x)} \\
 f_X(x) &= \int_{-\infty}^{\infty} f(x, y) dy \\
 &= \int_0^2 \left( \frac{x+y}{3} \right) dy = \frac{1}{3} \left[ xy + \frac{y^2}{2} \right]_0^2 \\
 &= \frac{2(x+1)}{3} \\
 f\left(\frac{Y}{X}\right) &= \frac{f(x, y)}{f_X(x)} = \frac{x+y}{2(x+1)}
 \end{aligned}$$

$$\begin{aligned}
 \text{Regression of } Y \text{ on } X &= E\left(\frac{Y}{X}\right) = \int_0^2 \frac{y(x+y)}{2(x+1)} dy \\
 &= \frac{1}{2(x+1)} \left[ \frac{xy^2}{2} + \frac{y^3}{3} \right]_0^2 \\
 &= \frac{1}{2(x+1)} \left[ 2x + \frac{8}{3} \right] = \frac{3x+4}{3(x+1)}
 \end{aligned}$$

$$\begin{aligned}
 E\left(\frac{X}{Y}\right) &= \int_{-\infty}^{\infty} xf\left(\frac{x}{y}\right) dx \\
 f\left(\frac{x}{y}\right) &= \frac{f(x,y)}{f_Y(y)} \\
 f_Y(y) &= \int_{-\infty}^{\infty} f(x,y) dx \\
 &= \int_0^1 \left( \frac{x+y}{3} \right) dx = \frac{1}{3} \left[ \frac{x^2}{2} + xy \right]_0^1 \\
 &= \frac{1}{3} \left[ \frac{1}{2} + y \right] \\
 f\left(\frac{x}{y}\right) &= \frac{2(x+y)}{2y+1}
 \end{aligned}$$

$$\begin{aligned}
 \text{Regression of } X \text{ on } Y &= E\left(\frac{X}{Y}\right) = \int_0^1 \frac{x+y}{2y+1} dx \\
 &= \frac{1}{2y+1} \left[ \frac{x^2}{2} + xy \right]_0^1 \\
 &= \frac{1}{2y+1} \left[ \frac{1}{2} + y \right] = \frac{1}{2}.
 \end{aligned}$$

b).

1. Correlation means relationship between two variables and Regression is a Mathematical Measure of expressing the average relationship between the two variables.
2. Correlation need not imply cause and effect relationship between the variables. Regression analysis clearly indicates the cause and effect relationship between Variables.
3. Correlation coefficient is symmetric i.e.  $r_{xy} = r_{yx}$  where regression coefficient is not symmetric
4. Correlation coefficient is the measure of the direction and degree of linear relationship between two variables. In regression using the relationship between two variables we can predict the dependent variable value for any given independent variable value.

**Problem 25.** a).  $X$  and  $Y$  are two random variables with variances  $\sigma_x^2$  and  $\sigma_y^2$  respectively and  $r$  is the coefficient of correlation between them. If  $U = X + KY$  and  $V = X + \frac{Y\sigma_x}{\sigma_y}$ , find the value of  $k$  so that  $U$  and  $V$  are uncorrelated.

b). Find the regression lines:

$X$	6	8	10	18	20	23
$Y$	40	36	20	14	10	2

**Solution:**

$$\text{Given } U = X + KY$$

$$E(U) = E(X) + KE(Y)$$

$$V = X + \frac{\sigma_x}{\sigma_y} Y$$

$$E(V) = E(X) + \frac{\sigma_x}{\sigma_y} E(Y)$$

If  $U$  and  $V$  are uncorrelated,  $\text{Cov}(U, V) = 0$

$$E[(U - E(U))(V - E(V))] = 0$$

$$\Rightarrow E \left[ (X + KY - E(X) - KE(Y)) \times \left( X + \frac{\sigma_x}{\sigma_y} Y - E(X) - \frac{\sigma_x}{\sigma_y} E(Y) \right) \right] = 0$$

$$\Rightarrow E \left\{ [(X - E(X)) + K(Y - E(Y))] \times [(X - E(X)) + \frac{\sigma_x}{\sigma_y} (Y - E(Y))] \right\} = 0$$

$$\Rightarrow E \left\{ (X - E(X))^2 + \frac{\sigma_x}{\sigma_y} (X - E(X))(Y - E(Y)) + K(Y - E(Y))(X - E(X)) + K \frac{\sigma_x}{\sigma_y} (Y - E(Y))^2 \right\} = 0$$

$$\Rightarrow V(X) + \frac{\sigma_x}{\sigma_y} Cov(X, Y) + KCov(X, Y) + K \frac{\sigma_x}{\sigma_y} V(Y) = 0$$

$$K \left[ Cov(X, Y) + \frac{\sigma_x}{\sigma_y} V(Y) \right] = -V(X) - \frac{\sigma_x}{\sigma_y} Cov(x, y)$$

$$K = \frac{-V(X) - \frac{\sigma_x}{\sigma_y} r \sigma_x \sigma_y}{r \sigma_x \sigma_y + \frac{\sigma_x}{\sigma_y} V(Y)} = \frac{-\sigma_x^2 - r \sigma_x^2}{r \sigma_x \sigma_y + \sigma_x \sigma_y}$$

$$= \frac{-\sigma_x^2 (1+r)}{\sigma_x \sigma_y (1+r)} = -\frac{\sigma_x}{\sigma_y}.$$

b).

$X$	$Y$	$X^2$	$Y^2$	$XY$
6	40	36	1600	240
8	36	64	1296	288
10	20	100	400	200
18	14	324	196	252
20	10	400	100	200
23	2	529	4	46
$\sum X = 85$	$\sum Y = 122$	$\sum X^2 = 1453$	$\sum Y^2 = 3596$	$\sum XY = 1226$

$$\bar{X} = \frac{\sum x}{n} = \frac{85}{6} = 14.17, \quad \bar{Y} = \frac{\sum y}{n} = \frac{122}{6} = 20.33$$

$$\sigma_x = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{1453}{6} - \left(\frac{85}{6}\right)^2} = 6.44$$

$$\sigma_y = \sqrt{\frac{\sum y^2}{n} - \left(\frac{\sum y}{n}\right)^2} = \sqrt{\frac{3596}{6} - \left(\frac{122}{6}\right)^2} = 13.63$$

$$r = \frac{\frac{\sum xy - \bar{x}\bar{y}}{\sigma_x\sigma_y}}{\frac{1226}{6} - (14.17)(20.33)} = \frac{6.44}{(6.44)(13.63)} = -0.95$$

$$b_{xy} = r \frac{\sigma_x}{\sigma_y} = -0.95 \times \frac{6.44}{13.63} = -0.45$$

$$b_{yx} = r \frac{\sigma_y}{\sigma_x} = -0.95 \times \frac{13.63}{6.44} = -2.01$$

The regression line  $X$  on  $Y$  is

$$x - \bar{x} = b_{xy}(y - \bar{y}) \Rightarrow x - 14.17 = -0.45(y - 20.33)$$

$$\Rightarrow x = -0.45y + 23.32$$

The regression line  $Y$  on  $X$  is

$$y - \bar{y} = b_{yx}(x - \bar{x}) \Rightarrow y - 20.33 = -2.01(x - 14.17)$$

$$\Rightarrow y = -2.01x + 48.81$$

**Problem 26.** a) Using the given information given below compute  $\bar{x}, \bar{y}$  and  $r$ . Also compute  $\sigma_y$  when  $\sigma_x = 2$ ,  $2x + 3y = 8$  and  $4x + y = 10$ .

b) The joint pdf of  $X$  and  $Y$  is

$X \backslash Y$	-1	1
0	$\frac{1}{8}$	$\frac{3}{8}$
1	$\frac{2}{8}$	$\frac{2}{8}$

Find the correlation coefficient of  $X$  and  $Y$ .

**Solution:**

a). When the regression equation are Known the arithmetic means are computed by solving the equation.

$$2x + 3y = 8 \quad \dots \dots \dots (1)$$

$$4x + y = 10 \quad \dots \dots \dots (2)$$

$$(1) \times 2 \Rightarrow 4x + 6y = 16 \quad \dots \dots \dots (3)$$

$$(2) - (3) \Rightarrow -5y = -6$$

$$\Rightarrow y = \frac{6}{5}$$

$$\text{Equation (1)} \Rightarrow 2x + 3\left(\frac{6}{5}\right) = 8$$

$$\Rightarrow 2x = 8 - \frac{18}{5}$$

$$\Rightarrow x = \frac{11}{5}$$

$$\text{i.e. } \bar{x} = \frac{11}{5} \text{ & } \bar{y} = \frac{6}{5}$$

To find  $r$ , Let  $2x + 3y = 8$  be the regression equation of  $X$  on  $Y$ .

$$2x = 8 - 3y \Rightarrow x = 4 - \frac{3}{2}y$$

$$\Rightarrow b_{xy} = \text{Coefficient of } Y \text{ in the equation of } X \text{ on } Y = -\frac{3}{2}$$

Let  $4x + y = 10$  be the regression equation of  $Y$  on  $X$

$$\Rightarrow y = 10 - 4x$$

$$\Rightarrow b_{yx} = \text{coefficient of } X \text{ in the equation of } Y \text{ on } X = -4.$$

$$\begin{aligned} r &= \pm \sqrt{b_{xy} b_{yx}} \\ &= -\sqrt{\left(-\frac{3}{2}\right)(-4)} \quad (\because b_{xy} \text{ & } b_{yx} \text{ are negative }) \\ &= -2.45 \end{aligned}$$

Since  $r$  is not in the range of  $(-1 \leq r \leq 1)$  the assumption is wrong.

Now let equation (1) be the equation of  $Y$  on  $X$

$$\Rightarrow y = \frac{8}{3} - \frac{2x}{3}$$

$\Rightarrow b_{yx}$  = Coefficient of  $X$  in the equation of  $Y$  on  $X$

$$b_{yx} = -\frac{2}{3}$$

from equation (2) be the equation of  $X$  on  $Y$

$$b_{xy} = -\frac{1}{4}$$

$$r = \pm \sqrt{b_{xy} b_{yx}} = \sqrt{-\frac{2}{3} \times -\frac{1}{4}} = 0.4081$$

To compute  $\sigma_y$  from equation (4)  $b_{yx} = -\frac{2}{3}$

$$\text{But we know that } b_{yx} = r \frac{\sigma_y}{\sigma_x}$$

$$\Rightarrow -\frac{2}{3} = 0.4081 \times \frac{\sigma_y}{2}$$

$$\Rightarrow \sigma_y = -3.26$$

b). Marginal probability mass function of  $X$  is

$$\text{When } X = 0, P(X) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$$

$$X = 1, P(X) = \frac{2}{8} + \frac{2}{8} = \frac{4}{8}$$

Marginal probability mass function of  $Y$  is

$$\text{When } Y = -1, P(Y) = \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$

$$Y = 1, P(Y) = \frac{3}{8} + \frac{2}{8} = \frac{5}{8}$$

$$E(X) = \sum_x x p(x) = 0 \times \frac{4}{8} + 1 \times \frac{4}{8} = \frac{4}{8}$$

$$E(Y) = \sum_y y p(y) = -1 \times \frac{3}{8} + 1 \times \frac{5}{8} = -\frac{3}{8} + \frac{5}{8} = \frac{2}{8}$$

$$E(X^2) = \sum_x x^2 p(x) = 0^2 \times \frac{4}{8} + 1^2 \times \frac{4}{8} = \frac{4}{8}$$

$$E(Y^2) = \sum_y y^2 p(y) = (-1)^2 \times \frac{3}{8} + 1^2 \times \frac{5}{8} = \frac{3}{8} + \frac{5}{8} = 1$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{4}{8} - \left( \frac{4}{8} \right)^2 = \frac{1}{4}$$

$$V(Y) = E(Y^2) - (E(Y))^2$$

$$= 1 - \left(\frac{1}{4}\right)^2 = \frac{15}{16}$$

$$E(XY) = \sum_x \sum_y xy p(x,y)$$

$$= 0 \times \frac{1}{8} + 0 \times \frac{3}{8} + (-1) \frac{2}{8} + 1 \times \left(\frac{2}{8}\right) = 0$$

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 0 - \frac{1}{2} \times \frac{1}{4} = -\frac{1}{8}$$

$$r = \frac{Cov(X, Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{-\frac{1}{8}}{\sqrt{\frac{1}{4}}\sqrt{\frac{15}{16}}} = -0.26.$$

**Problem 27.** a) Calculate the correlation coefficient for the following heights (in inches) of fathers  $X$  and their sons  $Y$ .

$X$	65	66	67	67	68	69	70	72
$Y$	67	68	65	68	72	72	69	71

b) If  $X$  and  $Y$  are independent exponential variates with parameters 1, find the pdf of  $U = X - Y$ .

**Solution:**

$X$	$Y$	$XY$	$X^2$	$Y^2$
65	67	4355	4225	4489
66	68	4488	4359	4624
67	65	4355	4489	4285
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041
$\sum X = 544$	$\sum Y = 552$	$\sum XY = 37560$	$\sum X^2 = 37028$	$\sum Y^2 = 38132$

$$\bar{X} = \frac{\sum x}{n} = \frac{544}{8} = 68$$

$$\bar{Y} = \frac{\sum y}{n} = \frac{552}{8} = 69$$

$$\bar{XY} = 68 \times 69 = 4692$$

$$\sigma_x = \sqrt{\frac{1}{n} \sum x^2 - \bar{X}^2} = \sqrt{\frac{1}{8} (37028) - 68^2} = \sqrt{4628.5 - 4624} = 2.121$$

$$\sigma_y = \sqrt{\frac{1}{n} \sum y^2 - \bar{y}^2} = \sqrt{\frac{1}{8} (38132) - 69^2} = \sqrt{4766.5 - 4761} = 2.345$$

$$\begin{aligned} \text{Cov}(X, Y) &= \frac{1}{n} \sum XY - \bar{X} \bar{Y} = \frac{1}{8} (37650) - 68 \times 69 \\ &= 4695 - 4692 = 3 \end{aligned}$$

The correlation coefficient of  $X$  and  $Y$  is given by

$$\begin{aligned} r(X, Y) &= \frac{\text{Cov}(X, Y)}{\sigma_x \sigma_y} = \frac{3}{(2.121)(2.345)} \\ &= \frac{3}{4.973} = 0.6032. \end{aligned}$$

b). Given that  $X$  and  $Y$  are exponential variates with parameters 1

$$f_X(x) = e^{-x}, x \geq 0, f_Y(y) = e^{-y}, y \geq 0$$

Also  $f_{XY}(x, y) = f_X(x) f_Y(y)$  since  $X$  and  $Y$  are independent

$$\begin{aligned} &= e^{-x} e^{-y} \\ &= e^{-(x+y)}; x \geq 0, y \geq 0 \end{aligned}$$

Consider the transformations  $u = x - y$  and  $v = y$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

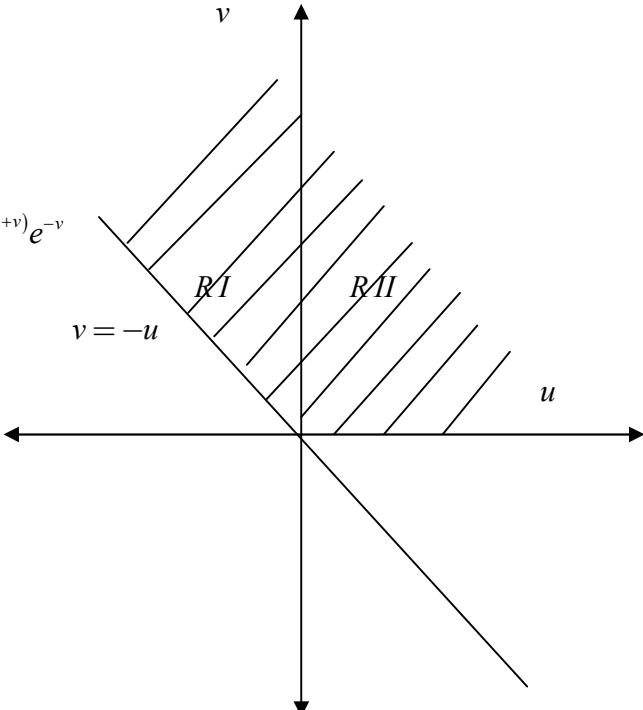
$$\begin{aligned} f_{UV}(u, v) &= f_{XY}(x, y) |J| = e^{-x} e^{-y} = e^{-(u+v)} e^{-v} \\ &= e^{-(u+2v)}, u + v \geq 0, v \geq 0 \end{aligned}$$

In Region I when  $u < 0$

$$\begin{aligned} f(u) &= \int_{-u}^{\infty} f(u, v) dv = \int_{-u}^{\infty} e^{-u} \cdot e^{-2v} dv \\ &= e^{-u} \left[ \frac{e^{-2v}}{-2} \right]_{-u}^{\infty} \\ &= \frac{e^{-u}}{-2} [0 - e^{2u}] = \frac{e^u}{2} \end{aligned}$$

In Region II when  $u > 0$

$$\begin{aligned} f(u) &= \int_0^{\infty} f(u, v) dv \\ &= \int_0^{\infty} e^{-(u+2v)} dv = \frac{e^{-u}}{2} \end{aligned}$$



$$\therefore f(u) = \begin{cases} \frac{e^u}{2}, & u < 0 \\ \frac{e^{-u}}{2}, & u > 0 \end{cases}$$

**Problem 28.** a) The joint pdf of  $X$  and  $Y$  is given by  $f(x, y) = e^{-(x+y)}, x > 0, y > 0$ . Find the pdf of  $U = \frac{X+Y}{2}$ .

b) If  $X$  and  $Y$  are independent random variables each following  $N(0, 2)$ , find the pdf of  $Z = 2X + 3Y$ . If  $X$  and  $Y$  are independent rectangular variates on  $(0, 1)$  find the distribution of  $\frac{X}{Y}$ .

**Solution:**

a). Consider the transformation  $u = \frac{x+y}{2}$  &  $v = y$   
 $\Rightarrow x = 2u - v$  and  $y = v$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$$\begin{aligned} f_{UV}(u, v) &= f_{XY}(x, y)|J| \\ &= e^{-(x+y)} 2 = 2e^{-(x+y)} = 2e^{-(2u-v+v)} \\ &= 2e^{-2u}, \quad 2u - v \geq 0, \quad v \geq 0 \end{aligned}$$

$$f_{UV}(u, v) = 2e^{-2u}, \quad u \geq 0, \quad 0 \leq v \leq \frac{u}{2}$$

$$f(u) = \int_0^{\frac{u}{2}} f_{UV}(u, v) dv = \int_0^{\frac{u}{2}} 2e^{-2u} dv$$

$$\begin{aligned} &= \left[ 2e^{-2u} v \right]_0^{\frac{u}{2}} \\ f(u) &= \begin{cases} 2 \frac{u}{2} e^{-2u}, & u \geq 0 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

b).(i) Consider the transformations  $w = y$ ,  
 i.e.  $z = 2x + 3y$  and  $w = y$

$$\text{i.e. } x = \frac{1}{2}(z - 3w), \quad y = w$$

$$|J| = \frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}.$$

Given that  $X$  and  $Y$  are independent random variables following  $N(0, 2)$

$$\therefore f_{XY}(x, y) = \frac{1}{8\pi} e^{-\frac{(x^2+y^2)}{8}}, -\infty < x, y < \infty$$

The joint pdf of  $(z, w)$  is given by

$$\begin{aligned} f_{ZW}(z, w) &= |J| f_{XY}(x, y) \\ &= \frac{1}{2} \cdot \frac{1}{8\pi} e^{-\frac{\left[\frac{1}{4}(z-3w)^2+w^2\right]}{8}} \\ &= \frac{1}{16\pi} e^{-\frac{1}{32}\left[(z-3w)^2+4w^2\right]}, -\infty < z, w < \infty. \end{aligned}$$

The pdf of  $z$  is the marginal pdf obtained by interchanging  $f_{ZW}(z, w)$  w.r.to  $w$  over the range of  $w$ .

$$\begin{aligned} \therefore f_z(z) &= \frac{1}{16\pi} \int_{-\infty}^{\infty} \left( e^{-\frac{1}{32}(z^2-6wz+13w^2)} \right) dw \\ &= \frac{1}{16\pi} e^{-\frac{z^2}{32}} \int_{-\infty}^{\infty} \left( e^{-\frac{13}{32}\left(w^2-\frac{6wz}{13}+\left(\frac{3z}{13}\right)^2-\left(\frac{3z}{13}\right)^2\right)} \right) dw \\ &= \frac{1}{16\pi} e^{-\frac{z^2}{32} + \frac{9z^2}{13 \times 32}} \int_{-\infty}^{\infty} \left( e^{-\frac{13}{32}\left(w-\frac{3z}{13}\right)^2} \right) dw = \frac{1}{16\pi} e^{-\frac{z^2}{8 \times 13}} \int_{-\infty}^{\infty} e^{-\frac{13}{32}t^2} dt \end{aligned}$$

$$r = \frac{13}{32}t^2 \Rightarrow dr = \frac{13}{16}tdt \Rightarrow \frac{16}{13t}dr = dt \Rightarrow \sqrt{\frac{r}{13}}dr = dt$$

$$\begin{aligned} \frac{16}{13} \sqrt{\frac{13}{r32}} dr &= dt \Rightarrow \frac{4}{\sqrt{13} \times \sqrt{2}} r^{-\frac{1}{2}} dr = dt \\ &= \frac{2}{16\pi} \frac{4}{\sqrt{13} \times \sqrt{2}} e^{-\frac{z^2}{8 \times 13}} \int_0^{\infty} e^{-r} r^{-\frac{1}{2}} dr \\ &= \frac{1}{2\pi\sqrt{13} \times \sqrt{2}} e^{-\frac{z^2}{8 \times 13}} \int_0^{\infty} e^{-r} r^{-\frac{1}{2}} dr = \frac{1}{2\pi\sqrt{13} \times \sqrt{2}} e^{-\frac{z^2}{8 \times 13}} \sqrt{\pi} = \frac{1}{2\sqrt{13}\sqrt{2\pi}} e^{-\frac{z^2}{2(2\sqrt{13})^2}} \end{aligned}$$

i.e.  $Z \sim N(0, 2\sqrt{13})$

b).(ii) Given that  $X$  and  $Y$  are uniform Variants over  $(0, 1)$

$$\therefore f_x(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \text{ and } f_y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Since  $X$  and  $Y$  are independent,

$$f_{XY}(x,y) = f_X(x)f_Y(y) \begin{cases} 1, & 0 < x, y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Consider the transformation  $u = \frac{x}{y}$  and  $v = y$

i.e.  $x = uv$  and  $y = v$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix} = v$$

$$\therefore f_{UV}(u,v) = f_{XY}(x,y)|J| \\ = v, \quad 0 < u < \infty, \quad 0 < v < \infty$$

The range for  $u$  and  $v$  are identified as follows.

$0 < x < 1$  and  $0 < y < 1$ .

$\Rightarrow 0 < uv < 1$  and  $0 < v < 1$

$\Rightarrow uv > 0, uv < 1, v > 0$  and  $v < 1$

$\Rightarrow uv > 0$  and  $v > 0 \Rightarrow u > 0$

$$\text{Now } f(u) = \int f_{UV}(u,v)dv$$

The range for  $v$  differs in two regions

$$f(u) = \int_0^1 f_{UV}(u,v)dv \\ = \int_0^1 vdv = \left[ \frac{v^2}{2} \right]_0^1 = \frac{1}{2}, \quad 0 < u < 1$$

$$f(u) = \int_0^{\frac{1}{u}} f_{UV}(u,v)dv = \int_0^{\frac{1}{u}} vdv = \left[ \frac{v^2}{2} \right]_0^{\frac{1}{u}} = \frac{1}{2u^2}, \quad 1 \leq u \leq \infty$$

$$\therefore f(u) = \begin{cases} \frac{1}{2}, & 0 \leq u \leq 1 \\ \frac{1}{2u^2}, & u > 1 \end{cases}$$

**Problem 29. a)** If  $X_1, X_2, \dots, X_n$  are Poisson variates with parameter  $\lambda = 2$ . Use the central limit theorem to estimate  $P(120 < S_n < 160)$  where  $S_n = X_1 + X_2 + \dots + X_n$  and  $n = 75$ .

b) A random sample of size 100 is taken from a population whose mean is 60 and variance is 400. Using central limit theorem, with what probability can we assert that the mean of the sample will not differ from  $\mu = 60$  by more than 4.

**Solution:**

a). Given that  $E(X_i) = \lambda = 2$  and  $Var(X_i) = \lambda = 2$

[Since in Poisson distribution mean and variance are equal to  $\lambda$  ]

i.e.  $\mu = 2$  and  $\sigma^2 = 2$

By central limit theorem,  $S_n \sim N(n\mu, n\sigma^2)$

$$S_n \sim N(150, 150)$$

$$\begin{aligned} \therefore P(120 < S_n < 160) &= P\left(\frac{120-150}{\sqrt{150}} < z < \frac{160-150}{\sqrt{150}}\right) \\ &= P(-2.45 < z < 0.85) \\ &= P(-2.45 < z < 0) + P(0 < z < 0.85) \\ &= P(0 < z < 2.45) + P(0 < z < 0.85) = 0.4927 + 0.2939 = 0.7866 \end{aligned}$$

b). Given that  $n = 100$ ,  $\mu = 60$ ,  $\sigma^2 = 400$

Since the probability statement is with respect to mean, we use the Linderberg-Levy form of central limit Theorem.

$$\begin{aligned} \bar{X} &\sim N\left(\mu, \frac{\sigma^2}{n}\right) \text{ i.e. } \bar{X} \text{ follows normal distribution with mean '}\mu\text{' and variance }\frac{\sigma^2}{n}. \\ \text{i.e. } \bar{X} &\sim N\left(60, \frac{400}{100}\right) \\ \bar{X} &\sim N(60, 4) \end{aligned}$$

$$\begin{aligned} P\left[\begin{array}{l} \text{mean of the sample will not} \\ \text{differ from 60 by more than 4} \end{array}\right] &= P\left[\begin{array}{l} \bar{X} \text{ will not differ from} \\ \mu = 60 \text{ by more than 4} \end{array}\right] \\ &= P(\bar{X} - \mu \leq 4) \\ &= P[-4 \leq \bar{X} - \mu \leq 4] \\ &= P[-4 \leq \bar{X} - 60 \leq 4] \\ &= P[56 \leq \bar{X} \leq 64] = P\left[\frac{56-60}{2} \leq z \leq \frac{64-60}{2}\right] \\ &= P[-2 \leq Z \leq 2] \\ &= 2P[0 \leq Z \leq 2] = 2 \times 0.4773 = 0.9446 \end{aligned}$$

**Problem 30** a) If the variable  $X_1, X_2, X_3, X_4$  are independent uniform variates in the interval  $(450, 550)$ , find  $P(1900 \leq X_1 + X_2 + X_3 + X_4 \leq 2100)$  using central limit theorem.

b) A distribution with unknown mean  $\mu$  has a variance equal to 1.5. Use central limit theorem to find how large a sample should be taken from the distribution in order that the probability will be at least 0.95 that the sample mean will be within 0.5 of the population mean.

### Solution

a). Given that  $X$  follows a uniform distribution in the interval  $(450, 550)$

$$\text{Mean} = \frac{b+a}{2} = \frac{450+550}{2} = 500$$

$$\text{Variance} = \frac{(b-a)^2}{12} = \frac{(550-450)^2}{12} = 833.33$$

By CLT  $S_n = X_1 + X_2 + X_3 + X_4$  follows a normal distribution with  $N(n\mu, n\sigma^2)$

The standard normal variable is given by  $Z = \frac{S_n - n\mu}{n\sigma}$

$$\text{when } S_n = 1900, Z = \frac{1900 - 4 \times 500}{\sqrt{4 \times 833.33}} = -\frac{100}{57.73} = -1.732$$

$$\text{when } S_n = 2100, Z = \frac{2100 - 2000}{\sqrt{4 \times 833.33}} = \frac{100}{57.73} = 1.732$$

$$\begin{aligned} \therefore P(1900 \leq S_n \leq 2100) &= P(-1.732 < z < 1.732) \\ &= 2 \times P(0 < z < 1.732) = 2 \times 0.4582 = 0.9164. \end{aligned}$$

b). Given  $E(X_i) = \mu$  and  $\text{Var}(X_i) = 1.5$

Let  $\bar{X}$  denote the sample mean.

By C.L.T.  $\bar{X}$  follows  $N\left(\mu, \frac{\sqrt{1.5}}{\sqrt{n}}\right)$

We have to find 'n' such that  $P(\mu - 0.5 < \bar{X} < \mu + 0.5) \geq 0.95$

i.e.  $P(-0.5 < \bar{X} - \mu < 0.5) \geq 0.95$

$$P(|\bar{X} - \mu| < 0.5) \geq 0.95$$

$$P\left[\left|z \frac{\sigma}{\sqrt{n}}\right| < 0.5\right] \geq 0.95$$

$$P\left[|z| < 0.5 \frac{\sqrt{n}}{\sigma}\right] \geq 0.95$$

$$P\left[|z| < 0.5 \frac{\sqrt{n}}{\sqrt{1.5}}\right] \geq 0.95$$

$$\text{ie } P(|Z| < 0.4082\sqrt{n}) \geq 0.95$$

Where 'Z' is the standard normal variable.

The Last value of 'n' is obtained from  $P(|Z| < 0.4082\sqrt{n}) = 0.95$

$$2P(0 < z < 0.4082\sqrt{n}) = 0.95$$

$$\Rightarrow 0.4082\sqrt{n} = 1.96 \Rightarrow n = 23.05$$

$\therefore$  The size of the sample must be atleast 24.