MISRIMAL NAVAJEE MUNOTH JAIN ENGINEERING COLLEGE, CHENNAI
DEPARTMENT OF MATHEMATICS
PROBABILITY AND RANDOM PROCESSES (MA2261)
SEMESTER –IV
UNIT-I: RANDOM VARIABLES
QUESTION BANK ANSWERS

PART-A

Problem1. X and Y are independent random variables with variance 2 and 3. Find the variance of $3X+4Y$.

Solution:

$$V(3X+4Y) = 9Var(X) + 16Var(Y) + 24Cov(XY)$$
$$= 9\times 2 + 16\times 3 + 0 \quad (\therefore X \,\&\, Y \text{ are independent } \operatorname{cov}(XY)=0)$$
$$= 18 + 48 = 66.$$

Problem 2. A Continuous random variable X has a probability density function $F(x)=3x^2$; $0\le x\le 1$. Find 'a' such that $P(x\le a)=P(x>a)$

Solution:

We know that the total probability =1

Given $P(X\le a)=P(X>a)=K(say)$

Then $K+K=1$

$$K=\frac{1}{2}$$

ie $P(X\le a)=\frac{1}{2} \,\&\, P(X>a)=\frac{1}{2}$

Consider $P(X\le a)=\frac{1}{2}$

$$\text{i.e. } \int_0^a f(x)\,dx=\frac{1}{2}$$
$$\int_0^a 3x^2\,dx=\frac{1}{2}$$
$$3\left(\frac{x^3}{3}\right)_0^a=\frac{1}{2}$$
$$a^3=\frac{1}{2}$$
$$a=\left(\frac{1}{2}\right)^{1/3}.$$

Problem 3. A random variable X has the p.d.f $f(x)$ given by $f(x)=\begin{cases} Cxe^{-x}; & if\ x>0 \\ 0 & ; if\ x\le 0\end{cases}$

Find the value of C and cumulative density function of X.

Solution:

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{0}^{\infty} Cxe^{-x}dx = 1$$

$$C\left[x\left(-e^{-x}\right)-\left(e^{-x}\right)\right]_0^{\infty} = 1$$

$$C = 1$$

$$\therefore f(x)=\begin{cases} xe^{-x}; x>0 \\ 0 \quad ; x\le 0 \end{cases}$$

$$C.D.F\ F(x)=\int_0^x f(x)dt=\int_0^x te^{-t}dt=\left[-te^{-t}-e^{-t}\right]_0^x=-xe^{-x}-e^{-x}+1$$

$$= 1-(1+x)e^{-x}.$$

Problem 4. If a random variable X has the p.d.f $f(x)=\begin{cases} \frac{1}{2}(x+1); -1<x<1 \\ 0 \quad ; otherwise \end{cases}$.

Find the mean and variance of X.

Solution:

$$\text{Mean}=\int_{-1}^{1} xf(x)\,dx=\frac{1}{2}\int_{-1}^{1} x(x+1)\,dx=\frac{1}{2}\int_{-1}^{1}\left(x^2+x\right)dx$$

$$=\frac{1}{2}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)_{-1}^{1}=\frac{1}{3}$$

$$\mu_2'=\int_{-1}^{1} x^2 f(x)dx=\frac{1}{2}\int_{-1}^{1}\left(x^3+x^2\right)dx=\frac{1}{2}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{1}$$

$$=\frac{1}{2}\left[\frac{1}{4}+\frac{1}{3}-\frac{1}{4}+\frac{1}{3}\right]$$

$$=\frac{1}{2}.\frac{2}{3}=\frac{1}{3}$$

$$Variance=\mu_2'-\left(\mu_1'\right)^2$$

$$=\frac{1}{3}-\frac{1}{9}=\frac{3-1}{9}=\frac{2}{9}.$$

Problem 5. A random variable X has density function given by $f(x)=\begin{cases} 2e^{-2x}; x\ge 0 \\ 0 \quad ; x<0 \end{cases}$.

Find m.g.f

Solution:

$$M_X(t)=E\left(e^{tx}\right)=\int_0^\infty e^{tx}f(x)\,dx=\int_0^\infty e^{tx}2e^{-2x}dx$$

$$=2\int_0^\infty e^{(t-2)x}dx$$

$$=2\left[\frac{e^{(t-2)x}}{t-2}\right]_0^\infty=\frac{2}{2-t},t<2.$$

Problem 6. Criticise the following statement: "The mean of a Poisson distribution is 5 while the standard deviation is 4".

Solution: For a Poisson distribution mean and variance are same. Hence this statement is not true.

Problem 7. Comment the following: "The mean of a binomial distribution is 3 and variance is 4

Solution:

In binomial distribution, mean $>$ variance but Variance $<$ Mean

Since Variance $=4$ &Mean $=3$, the given statement is wrong.

Problem8. If X and Y are independent binomial variates $B\left(5,\frac{1}{2}\right)$ and $B\left(7,\frac{1}{2}\right)$ find $P[X+Y=3]$

Solution:

$X+Y$ is also a binomial variate with parameters $n_1+n_2=12$ & $p=\frac{1}{2}$

$$\therefore P[X+Y=3]=12C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^9$$

$$=\frac{55}{2^{10}}$$

Problem 9. If X is uniformly distributed with Mean 1 and Variance $\frac{4}{3}$, find $P[X>0]$

Solution:

If X is uniformly distributed over (a,b), then

$$E(X)=\frac{b+a}{2}\ and\ V(X)=\frac{(b-a)^2}{12}$$

$$\therefore \frac{b+a}{2}=1\Rightarrow a+b=2$$

$$\Rightarrow \frac{(b-a)^2}{12}=\frac{4}{3}\Rightarrow (b-a)^2=16$$

$\Rightarrow a+b=2$ & $b-a=4$ We get $b=3,a=-1$

$\therefore a=-1$& $b=3$ and probability density function of x is

$$f(x)=\begin{cases}\frac{1}{4};-1<x<3\\0\,;Otherwise\end{cases}$$

$$P[x<0]=\int_{-1}^{0}\frac{1}{4}dx=\frac{1}{4}[x]_{-1}^{0}=\frac{1}{4}.$$

Problem 10. State the memoryless property of geometric distribution.

Solution:

If X has a geometric distribution, then for any two positive integer $'m'$ and $'n'$ $P\left[X>m+n\big/X>m\right]=P[X>n]$.

Problem 11. X is a normal variate with $mean=30$ and $S.D=5$

Find the following $P[26\le X\le 40]$

Solution:

$$X\sim N\left(30,5^2\right)$$

$$\therefore \mu=30\ \&\ \sigma=5$$

Let $Z=\dfrac{X-\mu}{\sigma}$ be the standard normal variate

$$P[26\le X\le 40]=P\left[\frac{26-30}{5}\le Z\le\frac{40-30}{5}\right]$$

$$=P[-0.8\le Z\le 2]=P[-0.8\le Z\le 0]+P[0\le Z\le 2]$$

$$=P[0\le Z\ 0.8]+[0\le z\le 2]$$

$$=0.2881+0.4772=0.7653.$$

Problem 12. If X is a $N(2,3)$ Find $P\left[Y\ge\frac{3}{2}\right]$ where $Y+1=X$.

Solution:

$$P\left[Y\ge\frac{3}{2}\right]=P\left[X-1\ge\frac{3}{2}\right]$$

$$=P[X\ge 2.5]=P[Z\ge 0.17]$$

$$=0.5-P[0\le Z\le 0.17]$$

$$=0.5-0.0675=0.4325$$

Problem 13. If the probability is $\frac{1}{4}$ that a man will hit a target what is the chance that he will hit the target for the first time in the 7^{th} trial?

Solution:

The required probability is

$$P[FFFFFFS]=P(F)P(F)P(F)P(F)P(F)P(F)P(S)$$

$$=q^6p=\left(\frac{3}{4}\right)^6.\left(\frac{1}{4}\right)=0.0445.$$

Hence $p =$ Probability of hitting target and $q = 1 - p$.

Problem 14. A random variable X has an exponential distribution defined by p.d.f. $f(x) = e^{-x},\ 0 < x < \infty$. Find the density function of $Y = 3X + 5$.

Solution:

$$y = 3x + 5 \Rightarrow \frac{dy}{dx} = 3 \Rightarrow \frac{dx}{dy} = \frac{1}{3}$$

P.d.f of y $h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

$$h_Y(y) = \frac{1}{3}e^{-x}.$$

Using $x = \frac{y-5}{3}$ we get $h_Y(y) = \frac{1}{3}e^{-\left(\frac{y-5}{3}\right)}, y > 5 (\because x > 0 \Rightarrow y > 5)$

Problem 15. If X is a normal variable with zero mean and variance σ^2, Find the p.d.f of $y = e^{-x}$

Solution:

Given $Y = e^{-x}$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

$$h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2}(\log y)^2} \times \frac{1}{y}.$$

PART-B

Problem 16. A random variable X has the following probability function:
Values of X,

X	:	0	1	2	3	4	5	6	7
$P(X)$	:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) K, (ii) Evaluate $P(X < 6), P(X \ge 6)$ and $P(0 < X < 5)$

(iii). Determine the distribution function of X.

(iv). $P(1.5 < X < 4.5 / X > 2)$

(v). $E(3x-4), Var(3x-4)$

Solution(i):

Since $\sum_{x=0}^{7} P(X) = 1$,

$$K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$10K^2+9K-1=0$$

$$K = \frac{1}{10} \quad or \quad K = -1$$

As $P(X)$ cannot be negative $K = \frac{1}{10}$

Solution(ii):

$$P(X<6)=P(X=0)+P(X=1)+...+P(X=5)$$
$$=\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+...=\frac{81}{100}$$

Now $P(X\geq 6)=1-P(X<6)$

$$=1-\frac{81}{100}=\frac{19}{100}$$

Now $P(0<X<5)=P(X=1)+P(X=2)+P(X=3)=P(X=4)$

$$=K+2K+2K+3K$$
$$=8K=\frac{8}{10}=\frac{4}{5}.$$

Solution(iii):

The distribution of X is given by $F_X(x)$ defined by

$$F_X(x)=P(X\leq x)$$

X	:	0	1	2	3	4	5	6	7
$F_X(x)$	:	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{4}{5}$	$\frac{81}{100}$	$\frac{83}{100}$	1

Problem 17. (a) If $P(x)=\begin{cases}\frac{x}{15};x=1,2,3,4,5\\ 0\quad ;elsewhere\end{cases}$

Find (i) $P\{X=1or2\}$ and (ii) $P\{1/2<X<5/2/x>1\}$

(b) X is a continuous random variable with pdf given by

$$F(X)=\begin{cases}Kx & in\quad 0\leq x\leq 2\\ 2K & in\quad 2\leq x\leq 4\\ 6K-Kx & in\quad 4\leq x\leq 6\\ 0 & elsewhere\end{cases}$$

Find the value of K and also the cdf $F_X(x)$.

Solution:

(a) i) $P(X=1\,or\,2)=P(X=1)+P(X=2)$

$$=\frac{1}{15}+\frac{2}{15}=\frac{3}{15}=\frac{1}{5}$$

ii) $P\left(\frac{1}{2}<X<\frac{5}{2}/x>1\right)=\dfrac{P\left\{\left(\frac{1}{2}<X<\frac{5}{2}\right)\cap(X>1)\right\}}{P(X>1)}$

$$=\frac{P\{(X=1or2)\cap(X>1)\}}{P(X>1)}$$

$$= \frac{P(X=2)}{1-P(X=1)}$$

$$= \frac{2/15}{1-(1/15)} = \frac{2/15}{14/15} = \frac{2}{14} = \frac{1}{7}.$$

Since $\int_{\infty}^{\infty} F(x)\,dx = 1$

$$\int_0^2 Kxdx + \int_2^4 2Kdx + \int_4^6 (6k - kx)dx = 1$$

$$K\left[\left(\frac{x^2}{2}\right)_0^2 + (2x)_2^4 + \int_4^6 \left(6x - \frac{x^2}{2}\right)_4^6\right] = 1$$

$$K\left[\cancel{2} + \cancel{8} - 4 + 36 - 18 - 24 + 8\right] = 1$$

$$8K = 1 \quad K = \frac{1}{8}$$

We know that $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

If $x < 0$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx = 0$

If $x \in (0,2)$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

$$F_X(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^x f(x)\,dx$$

$$= \int_{-\infty}^{0} 0dx + \int_0^x Kxdx = \int_{-\infty}^{0} 0dx + \frac{1}{8}\int_0^x xdx$$

$$= \left(\frac{x^2}{16}\right)_0^x = \frac{x^2}{16}, 0 \le x \le 2$$

If $x \in (2,4)$, then $F_X(x) = \int_{-\infty}^{x} f(x)\,dx$

$$F_X(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^2 f(x)\,dx + \int_2^x f(x)\,dx$$

$$= \int_{-\infty}^{0} 0dx + \int_0^2 Kxdx + \int_2^x 2Kdx$$

$$= \int_0^2 \frac{x}{8}dx + \int_2^x \frac{1}{4}dx = \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^x$$

$$= \frac{1}{4} + \frac{x}{4} - \frac{1}{2}$$

$$= \frac{x}{4} - \frac{4}{16} = \frac{x-1}{4}, 2 \le x < 4$$

If $x \in (4,6)$, then $F_X(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{2} Kxdx + \int_{2}^{4} 2Kdx + \int_{4}^{x} k(6-x)dx$

$$= \int_{0}^{2} \frac{x}{8}dx + \int_{2}^{4} \frac{1}{4}dx + \int_{4}^{x} \frac{1}{8}(6-x)dx$$

$$= \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^4 + \left(\frac{6x}{8} - \frac{x^2}{16}\right)_4^x$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \frac{6x}{8} - \frac{x^2}{16} - 3 + 1$$

$$= \frac{4+16-8+12x-x^2-48+16}{16}$$

$$= \frac{-x^2+12x-20}{16}, 4 \le x \le 6$$

If $x > 6$, then $F_X(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{2} Kxdx + \int_{2}^{4} 2Kdx + \int_{4}^{6} k(6-x)dx + \int_{6}^{\infty} 0dx$

$$= 1, x \ge 6$$

$$\therefore F_X(x) = \begin{cases} 0 & ; x \le 0 \\ \dfrac{x^2}{16} & ; 0 \le x \le 2 \\ \dfrac{1}{4}(x-1) & ; 2 \le x \le 4 \\ \dfrac{-1}{16}(20-12x+x^2) & ; 4 \le x \le 6 \\ \quad 1 & ; x \ge 6 \end{cases}$$

Problem18. (a). A random variable X has density function $f(x) = \begin{cases} \dfrac{K}{1+x^2}, & -\infty < x < \infty \\ 0 \quad, & Otherwise \end{cases}$. Determine K and the distribution functions. Evaluate the probability $P(x \ge 0)$.

(b). A random variable X has the P.d.f $f(x) = \begin{cases} 2x, 0 < x < 1 \\ 0 \quad, Otherwise \end{cases}$

Find (i) $P\left(X < \frac{1}{2}\right)$ (ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right)$ (iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right)$

Solution (a):

$$\text{Since } \int_{-\infty}^{\infty} F(x)dx = 1$$

$$\int_{-\infty}^{\infty} \frac{K}{1+x^2}dx = 1$$

$$K\int_{\infty}^{\infty} \frac{dx}{1+x^2} = 1$$

$$K\left(\tan^{-1} x\right)_{-\infty}^{\infty} = 1$$

$$K\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

$$K\pi = 1$$

$$K = \frac{1}{\pi}$$

$$F_X(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{K}{1+x^2}dx$$

$$= \frac{1}{\pi}\left(\tan^{-1} x\right)_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1} x\right], -\infty < x < \infty$$

$$P(X \geq 0) = \frac{1}{\pi}\int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1} x\right)_{0}^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1} 0\right) = \frac{1}{2}.$$

Solution (b):

(i) $P\left(x < \frac{1}{2}\right) = \int_{0}^{1/2} f(x)\,dx = \int_{0}^{1/2} 2x dx = 2\left(\frac{x^2}{2}\right)_{0}^{1/2} = \frac{2 \times 1}{8} = \frac{1}{4}$

(ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 2x dx = 2\left(\frac{x^2}{2}\right)_{1/4}^{1/2}$

$$= 2\left(\frac{1}{8} - \frac{1}{32}\right) = \left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{16}.$$

(iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right) = \frac{P\left(X > \frac{3}{4} \cap X > \frac{1}{2}\right)}{P\left(X > \frac{1}{2}\right)} = \frac{P\left(X > \frac{3}{4}\right)}{P\left(X > \frac{1}{2}\right)}$

$$P\left(X>\frac{3}{4}\right)=\int_{3/4}^{1} f(x)\,dx=\int_{3/4}^{1} 2x\,dx=2\left(\frac{x^2}{2}\right)_{3/4}^{1}=1-\frac{9}{16}=\frac{7}{16}$$

$$P\left(X>\frac{1}{2}\right)=\int_{1/2}^{1} f(x)\,dx=\int_{1/2}^{1} 2x\,dx=2\left(\frac{x^2}{2}\right)_{1/2}^{1}=1-\frac{1}{4}=\frac{3}{4}$$

$$P\left(X>\frac{3}{4}/X>\frac{1}{2}\right)=\frac{\frac{7}{16}}{\frac{3}{4}}=\frac{7}{16}\times\frac{4}{3}=\frac{7}{12}.$$

Problem 19.(a).If X has the probability density function $f(x)=\begin{cases}Ke^{-3x}, & x>0\\ 0 & ,otherwise\end{cases}$ find K, $P[0.5\le X\le 1]$ and the mean of X.

(b).Find the moment generating function for the distribution whose p.d.f is $f(x)=\lambda e^{-\lambda x},\ x>0$ and hence find its mean and variance.

Solution:

Since $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_{0}^{\infty} Ke^{-3x}dx=1$$

$$K\left[\frac{e^{-3x}}{-3}\right]_0^{\infty}=1$$

$$\frac{K}{3}=1$$

$$K=3$$

$$P(0.5\le X\le 1)=\int_{0.5}^{1} f(x)\,dx=3\int_{0.5}^{1} e^{-3x}dx=\cancel{3}\,\frac{e^{-3}-e^{-1.5}}{-\cancel{3}}=\left[e^{-1.5}-e^{-3}\right]$$

Mean of $X=E(x)=\int_0^{\infty} xf(x)\,dx=3\int_0^{\infty} xe^{-3x}dx$

$$=3\left[x\left(\frac{-e^{-3x}}{3}\right)-1\left(\frac{e^{-3x}}{9}\right)\right]_0^{\infty}=\frac{3\times 1}{9}=\frac{1}{3}$$

Hence the mean of $X=E(X)=\frac{1}{3}$

$$M_X(t)=E\left(e^{tx}\right)=\int_0^{\infty} e^{tx}f(x)\,dx=\int_0^{\infty}\lambda e^{-\lambda x}e^{tx}dx$$

$$=\lambda\int_0^{\infty} e^{-x(\lambda-t)}dx$$

$$= \lambda \left[\frac{e^{-x(\lambda - t)}}{-(\lambda - t)} \right]_0^\infty = \frac{\lambda}{\lambda - t}$$

$$\text{Mean} = \mu_1' = \left[\frac{d}{dt} M_X(t) \right]_{t=0} = \left[\frac{\lambda}{(\lambda - t)^2} \right]_{t=0} = \frac{1}{\lambda}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t) \right]_{t=0} = \left[\frac{\lambda(2)}{(\lambda - t)^3} \right]_{t=0} = \frac{2}{\lambda^2}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Problem 20. (a). If the continuous random variable X has ray Leigh density $F(x) = \left(\frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} \right) \times U(x)$ find $E(x^n)$ and deduce the values of $E(X)$ and $Var(X)$.

(b). Let the random variable X have the p.d.f $f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & , x > 0 \\ 0 & , otherwise. \end{cases}$.

Find the moment generating function, mean & variance of X.

Solution:

(a) Here $U(x) = \begin{cases} 1 & if \quad x > 0 \\ 0 & if \quad x \le 0 \end{cases}$

$$E(x^n) = \int_0^\infty x^n f(x) dx$$

$$= \int_0^\infty x^n \frac{x}{\alpha^2} e^{\frac{-x^2}{2\alpha^2}} dx$$

Put $\frac{x^2}{2\alpha^2} = t$, $\qquad x = 0,\ t = 0$

$\frac{x}{\alpha^2} dx = dt \qquad x = \alpha, t = \infty$

$$= \int_0^\infty (2\alpha^2 t)^{n/2} e^{-t} dt \quad \left[\because x = \sqrt{2\alpha}.\sqrt{t} \right]$$

$$= 2^{n/2} \alpha^n \int_0^\infty t^{n/2} e^{-t} dt$$

$$E(x^n) = 2^{n/2} \alpha^n \Gamma\left(\frac{n}{2} + 1 \right) - (1)$$

Putting $n = 1$ in (1) we get

$$E(x) = 2^{1/2} \alpha\, \Gamma\left(\frac{3}{2} \right) = \sqrt{2} \alpha\, \Gamma\left(\frac{1}{2} + 1 \right)$$

$$= \sqrt{2}\alpha \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\alpha}{\sqrt{2}}\sqrt{\pi} \quad [\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\therefore E(x) = \alpha\sqrt{\frac{\pi}{2}}$$

Putting $n = 2$ in (1), we get

$$E(x^2) = 2\alpha^2\Gamma(2) = 2\alpha^2 \quad [\because \Gamma(2) = 1]$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2$$

$$= 2\alpha^2 - \alpha^2\frac{\pi}{2}$$

$$= \left(2 - \frac{\pi}{2}\right)\alpha^2 = \left(\frac{4-\pi}{2}\right)\alpha^2.$$

(b) $M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx}\frac{1}{2}e^{-x/2}dx$

$$= \frac{1}{2}\int_0^{\infty} e^{\left(t-\frac{1}{2}\right)x}dx = \frac{1}{2}\left[\frac{e^{\left(t-\frac{1}{2}\right)x}}{\left(t-\frac{1}{2}\right)}\right]_0^{\infty} = \frac{1}{1-2t},\ if\ t < \frac{1}{2}.$$

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{2}{(1-2t)^2}\right]_{t=0} = 2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{8}{(1-2t)^3}\right]_{t=0} = 8$$

$$Var(X) = E(X^2) - [E(X)]^2 = 8 - 4 = 4.$$

Problem 21. (a).The elementary probability law of a continues random variable is $f(x) = y_0 e^{-b(x-a)},\quad a \le x \le \infty,\ b > 0$ where a, b and y_0 are constants. Find y_0 the rth moment about point $x = a$ and also find the mean and variance.

(b).The first four moments of a distribution about $x = 4$ are 1,4,10 and 45 respectively. Show that the mean is 5, variance is 3, $\mu_3 = 0$ and $\mu_4 = 26$.

Solution:

Since the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$y_0\int_0^{\infty} e^{-b(x-a)}dx = 1$$

$$y_0\left[\frac{e^{-b(x-a)}}{-b}\right]_0^{\infty}=1$$

$$y_0\left(\frac{1}{b}\right)=1$$

$$y_0=b.$$

μ_r' (r$^{\text{th}}$ moment about the point $x=a$) $= \int_{-\infty}^{\infty}(x-a)^r f(x)\,dx$

$$=b\int_a^{\infty}(x-a)^r e^{-b(x-a)}dx$$

Put $x-a=t$, $dx=dt$, when $x=a, t=0$, $x=\infty, t=\infty$

$$=b\int_0^{\infty}t^r e^{-bt}dt$$

$$=b\frac{\Gamma(r+1)}{b^{(r+1)}}=\frac{r!}{b^r}$$

In particular $r=1$

$$\mu_1'=\frac{1}{b}$$

$$\mu_2'=\frac{2}{b^2}$$

Mean $=a+\mu_1'=a+\frac{1}{b}$

Variance $=\mu_2'-\left(\mu_1'\right)^2$

$$=\frac{2}{b^2}-\frac{1}{b^2}=\frac{1}{b^2}.$$

b) Given $\mu_1'=1, \mu_2'=4, \mu_3'=10, \mu_4'=45$

$\mu_r'=r^{th}$ moment about to value $x=4$

Here $A=4$

Here Mean $=A+\mu_1'=4+1=5$

Variance $=\mu_2=\mu_2'-\left(\mu_1'\right)^2$

$$=4-1=3.$$

$$\mu_3=\mu_3'-3\mu_2'\mu_1'+2\left(\mu_1'\right)^3$$

$$=10-3(4)(1)+2(1)^3=0$$

$$\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'\left(\mu_1'\right)^2-3\left(\mu_1'\right)^4$$

$$= 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4$$

$$\mu_4 = 26.$$

Problem 22. (a). A continuous random variable X has the p.d.f $f(x) = kx^2e^{-x},\ x \ge 0$. Find the r[th] moment of X about the origin. Hence find mean and variance of X.

(b). Find the moment generating function of the random variable X, with probability density function $f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 2 - x & \text{for } 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$. Also find μ_1', μ_2'.

Solution:

$$\text{Since } \int_0^\infty Kx^2e^{-x}dx = 1$$

$$K\left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty = 1$$

$$2K = 1$$

$$K = \frac{1}{2}.$$

$$\mu_r' = \int_0^\infty x^r f(x)dx$$

$$= \frac{1}{2}\int_0^\infty x^{r+2}e^{-x}dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}x^{(r+3)-1}dx = \frac{(r+2)!}{2}$$

Putting $n = 1,\ \mu_1' = \frac{3!}{2} = 3$

$$n = 2,\ \mu_2' = \frac{41}{2} = 12$$

$\therefore$ Mean $= \mu_1' = 3$

Variable$= \mu_2' - \left(\mu_1'\right)^2$

i.e. $\mu_2 = 12 - (3)^2 = 12 - 9$

$\therefore \mu_2 = 3.$

(b) $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

$$= \int_0^1 e^{tx}xdx + \int_1^2 e^{tx}(2 - x)\,dx$$

$$= \left(\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2} \right)_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2} \right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \left(\frac{e^t - 1}{t} \right)^2$$

$$= \left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots - 1 \right]^2$$

$$= \left[1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots \right]^2$$

$$\mu_1' = coeff.\ of\ \frac{t}{1!} = 1$$

$$\mu_2' = coeff.\ of\ \frac{t^2}{2!} = \frac{7}{6}.$$

Problem 23. (a).The p.d.f of the r.v. X follows the probability law: $f(x) = \frac{1}{2\theta} e^{-\frac{|x-\theta|}{\theta}}$, $-\infty < x < \infty$. Find the m.g.f of X and also find $E(X)$ and $V(X)$.

(b).Find the moment generating function and rth moments for the distribution. Whose p.d.f is $f(x) = Ke^{-x}$, $0 \le x \le \infty$. Find also standard deviation.

Solution:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{2\theta} e^{\frac{-|x-\theta|}{\theta}} e^{tx}\,dx$$

$$= \int_{-\infty}^{\theta} \frac{1}{2\theta} e^{\frac{(x-\theta)}{\theta}} e^{tx}\,dx + \int_{\theta}^{\infty} \frac{1}{2\theta} e^{\frac{-(x-\theta)}{\theta}} e^{tx}\,dx$$

$$M_X(t) = \frac{e^{-1}}{2\theta} \int_{-\infty}^{\theta} e^{x\left(t+\frac{1}{\theta}\right)}\,dx + \frac{e}{2\theta} \int_{\theta}^{\infty} e^{-x\left(\frac{1}{\theta}-t\right)}\,dx$$

$$= \frac{e^{-1}}{2\theta} \frac{e^{\theta\left(t+\frac{1}{\theta}\right)}}{\left(t+\frac{1}{\theta}\right)} + \frac{e}{2\theta} \frac{e^{-\theta\left(\frac{1}{\theta}-1\right)}}{\left(\frac{1}{\theta}-t\right)}$$

$$= \frac{e^{\theta t}}{2(\theta t+1)} + \frac{e^{\theta t}}{2(1-\theta t)} = \frac{e^{\theta t}}{1-\theta^2 t^2} = e^{\theta t}\left[1-(\theta t)^2\right]^{-1}$$

$$= \left[1 + \theta t + \frac{\theta^2 t^2}{2!} + \ldots \right]\left[1 + \theta^2 t^2 + \theta^4 t^4 + \ldots \right]$$

$$= 1 + \theta t + \frac{3\theta^2 t^2}{2!} + \ldots$$

$$E(X)=\mu_1'=coeff.\ of\ t\ in\ M_X(t)=\theta$$

$$\mu_2'=coeff.\ of\ \frac{t^2}{2!}\ in\ M_X(t)=3\theta^2$$

$$Var(X)=\mu_2'-\left(\mu_1'\right)^2=3\theta^2-\theta^2=2\theta^2.$$

b)

Total Probability=1

$$\therefore \int_0^\infty ke^{-x}dx=1$$

$$k\left[\frac{e^{-x}}{-1}\right]_0^\infty=1$$

$$k=1$$

$$M_X(t)=E\left[e^{tx}\right]=\int_0^\infty e^{tx}e^{-x}dx=\int_0^\infty e^{(t-1)x}dx$$

$$=\left[\frac{e^{(t-1)x}}{t-1}\right]_0^\infty=\frac{1}{1-t},\ t<1$$

$$=(1-t)^{-1}=1+t+t^2+\ldots+t^r+\ldots\infty$$

$$\mu_1'=coeff.\ of\ \frac{t^2}{r!}=r!$$

When $r=1$, $\mu_1'=1!=1$

$r=2$, $\mu_2'=2!=2$

Variance $=\mu_2'-\mu_1'=2-1=1$

$\therefore$ Standard deviation=1.

Problem 24. (a). Define Binomial distribution Obtain its m.g.f., mean and variance.

(b). (i).Six dice are thrown 729 times. How many times do you expect atleast 3 dice show 5 or 6 ?

(ii).Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads x times?

Solution:

a) A random variable X said to follow binomial distribution if it assumes only non negative values and its probability mass function is given by $P(X=x)=nC_x p^x q^{n-x},\ x=0,1,2,\ldots,n\ and\ q=1-p.$

M.G.F.of Binomial distribution:-

M.G.F of Binomial Distribution about origin is

$$M_X(t)=E\left[e^{tx}\right]=\sum_{x=0}^{n}e^{tx}P(X=x)$$

$$=\sum_{x=0}^{n}nC_x x\,P^x q^{n-x}e^{tx}$$

$$= \sum_{x=0}^{n} nC_x \left(pe^t\right)^x q^{n-x}$$

$$M_X(t) = \left(q + pe^t\right)^n$$

Mean of Binomial distribution

$$\text{Mean} = E(X) = M_X'(0)$$

$$= \left[n\left(q + pe^t\right)^{n-1} pe^t\right]_{t=0} = np \text{ Since } q + p = 1$$

$$E\left(X^2\right) = M_X''(0)$$

$$= \left[n(n-1)\left(q + pe^t\right)^{n-2}\left(pe^t\right)^2 + npe^t\left(q + pe^t\right)^{n-1}\right]_{t=0}$$

$$E\left(X^2\right) = n(n-1)p^2 + np$$

$$= n^2p^2 + np(1-p) = n^2p^2 + npq$$

$$\text{Variance} = E\left(X^2\right) - \left[E[X]\right]^2 = npq$$

$$Mean = np \,; Variance = npq$$

b) Let X : the number of times the dice shown 5 or 6

$$P[5 \text{ or } 6] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$\therefore P = \frac{1}{3} \text{ and } q = \frac{2}{3}$$

Here $n = 6$

To evaluate the frequency of $X \geq 3$

By Binomial theorem,

$$P[X = r] = 6C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{6-r} \text{ where } r = 0,1,2\ldots6.$$

$$P[X \geq 3] = P(3) + P(4) + P(5) + P(6)$$

$$= 6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + 6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + 6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + 6C_6\left(\frac{1}{3}\right)^6$$

$$= 0.3196$$

$\therefore$ Expected number of times atleast 3 dies to show 5 or 6 $= N \times P[X \geq 3]$

$$= 729 \times 0.3196 = 233.$$

25. (a). A die is cast until 6 appears what is the probability that it must cast more then five times?

(b). Suppose that a trainee soldier shoots a target an independent fashion. If the probability that the target is shot on any one shot is 0.8.

(i) What is the probability that the target would be hit on 6^{th} attempt?

(ii) What is the probability that it takes him less than 5 shots?

Solution:

Probability of getting six $=\frac{1}{6}$

$\therefore p=\frac{1}{6}$ & $q=1-\frac{1}{6}$

Let x: No of throws for getting the number 6.By geometric distribution $P[X=x]=q^{x-1}p, x=1,2,3....$

Since 6 can be got either in first, second……throws.

To find $P[X>5]=1-P[X\le 5]$

$$=1-\sum_{x=1}^{5}\left(\frac{5}{6}\right)^{x-1}.\frac{1}{6}$$

$$=1-\left[\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)\right]$$

$$=1-\frac{\frac{1}{6}\left[1-\left(\frac{5}{6}\right)^5\right]}{1-\frac{5}{6}}=\left(\frac{5}{6}\right)^5=0.4019$$

b) Here $p=0.8, q=1-p=0.2$

$$P[X=r]=q^{r-1}p, r=0,1,2...$$

(i) The probability that the target would be hit on the 6th attempt $=P[X=6]$

$$=(0.2)^5(0.8)=0.00026$$

(ii) The probability that it takes him less than 5 shots $=P[X<5]$

$$=\sum_{r=1}^{4}q^{r-1}p=0.8\sum_{r=1}^{4}(0.2)^{r-1}$$

$$=0.8[1+0.2+0.04+0.008]=0.9984$$

Problem 26. (a). State and prove the memoryless property of exponential distribution.

(b). A component has an exponential time to failure distribution with mean of 10,000 hours.

(i). The component has already been in operation for its mean life. What is the probability that it will fail by 15,000 hours?

(ii). At 15,000 hours the component is still in operation. What is the probability that it will operate for another 5000 hours.

Solution:

a) Statement:

If X is exponentially distributed with parameters λ, then for any two positive integers‘s’ and‘t’, $P[x>s+t/x>s]=P[x>t]$

Proof:

The p.d.f of X is $f(x)=\begin{cases}\lambda e^{-\lambda x}, x\ge 0\\ 0 \quad ,Otherwise\end{cases}$

$$\therefore P[X>t]=\int_t^{\infty}\lambda e^{-\lambda x}dx=\left[-e^{-\lambda x}\right]_t^{\infty}=e^{-\lambda t}$$

$$\therefore P[X>s+t/x>s]=\frac{P[x>s+t\cap x>s]}{P[x>s]}$$

$$=\frac{P[X>s+t]}{P[X>s]}=\frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}=e^{-\lambda t}$$

$$=P[x>t]$$

b) Let X denote the time to failure of the component then X has exponential distribution with $Mean=1000$ hours.

$$\therefore\frac{1}{\lambda}=10,000\Rightarrow\lambda=\frac{1}{10,000}$$

The p.d.f. of X is $f(x)=\begin{cases}\dfrac{1}{10,000}e^{-\frac{x}{10,000}} & ,x\geq 0\\ 0 & ,otherwise\end{cases}$

(i) Probability that the component will fail by 15,000 hours given it has already been in operation for its mean life $=P[x<15,000/x>10,000]$

$$=\frac{P[10,000<X<15,000]}{P[X>10,000]}$$

$$=\frac{\int_{10,000}^{15,000}f(x)dx}{\int_{10,000}^{\infty}f(x)dx}=\frac{e^{-1}-e^{-1.5}}{e^{-1}}$$

$$=\frac{0.3679-0.2231}{0.3679}=0.3936.$$

(ii) Probability that the component will operate for another 5000 hours given that it is in operational 15,000 hours $=P[X>20,000/X>15,000]$

$$=P[x>5000]\ \text{[By memoryless prop]}$$

$$=\int_{5000}^{\infty}f(x)dx=e^{-0.5}=0.6065$$

27. (a). The Daily consumption of milk in a city in excess of 20,000 gallons is approximately distributed as a Gamma variate with parameters $\alpha=2$ and $\lambda=\frac{1}{10,000}$. The city has a daily stock of 30,000 gallons. What is the probability that the stock is in sufficient on a particular day?

(b). The lifetime (in hours) of a certain piece of equipment is a continuous r.v. having range $0 < x < \infty$ and p.d.f.is $f(x) = \begin{cases} xe^{-kx}, 0 < x < \infty \\ 0 \quad , otherwise \end{cases}$. Determine the constant K and evaluate the probability that the life time exceeds 2 hours.

Solution:

a) Let X be the r.v denoting the daily consumption of milk (is gallons) in a city

Then $Y = X - 20,000$ has Gamma distribution with p.d.f.

$$f(y) = \frac{1}{(10,000)^2 \Gamma(2)} y^{2-1} e^{-\frac{y}{10,000}}, y \geq 0$$

$$f(y) = \frac{ye^{-\frac{y}{10,000}}}{(10,000)^2}, y \geq 0.$$

$\therefore$ the daily stock of the city is 30,000 gallons, the required probability that the stock is insufficient on a particular day is given by

$$P[X > 30,000] = P[Y > 10,000]$$

$$= \int_{10,000}^{\infty} g(y)\,dy = \int_{10,000}^{\infty} \frac{ye^{-\frac{y}{10,000}}}{(10,000)^2} dy$$

put $Z = \frac{y}{10,000}$ then $dz = \frac{dy}{10,000}$

$$\therefore P[X > 30,000] = \int_1^{\infty} ze^{-z} dz$$

$$= \left[-ze^{-z} - e^{-z}\right]_1^{\infty} = \frac{2}{e}$$

b) Let X the life time of a certain piece of equipment.

Then the p.d.f. $f(x) = \begin{cases} xe^{-kx}, 0 < x < \infty \\ 0 \quad , Otherwise \end{cases}$

To find K, $\int_0^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} e^{-kx} x^{2-1} dx = 1$$

$$\frac{\Gamma(2)}{K^2} = 1 \Rightarrow K^2 = 1 \quad \therefore K = 1$$

$$\therefore f(x) = \begin{cases} xe^{-x}, 0 < x < \infty \\ 0 \quad , Otherwise \end{cases}$$

P[Life time exceeds 2 hours] $= P[X > 2]$

$$= \int_2^{\infty} f(x)\,dx$$

$$= \int_{2}^{\infty} x e^{-x} dx$$

$$= \left[x\left(-e^{-x}\right) - \left(e^{-x}\right) \right]_{2}^{\infty}$$

$$= 2e^{-2} + e^{-2} = 3e^{-2} = 0.4060$$

Problem 28. (a). State and prove the additive property of normal distribution.

(b). Prove that "For standard normal distribution $N(0,1)$, $M_X(t) = e^{\frac{t^2}{2}}$.

a) Statement:

If $X_1, X_2, ..., X_n$ are n independent normal random variates with mean (μ_1, σ_1^2), $(\mu_2, \sigma_2^2), \dots (\mu_n, \sigma_n^2)$ then $X_1 + X_2 + ... + X_n$ also a normal random variable with mean $\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2 \right)$.

Proof:

We know that. $M_{X_1+X_2+...+X_n}(t) = M_{X_1}(t) M_{X_2}(t) ... M_{X_n}(t)$

But $M_{X_i}(t) = e^{\mu_i t + \frac{t^2 \sigma_i^2}{2}}$, $i = 1, 2....n$

$$M_{X_1+X_2+...+X_n}(t) = e^{\mu_1 t + \frac{t^2\sigma_1^2}{2}} e^{\mu_2 t + \frac{t^2\sigma_2^2}{2}} ... e^{\mu_n t + \frac{t^2\sigma_n^2}{2}}$$

$$= e^{(\mu_1 + \mu_2 + ... + \mu_n)t + \frac{(\sigma_1^2 + \sigma_2^2 + ... + \sigma_n^2)t^2}{2}}$$

$$= e^{\sum_{i=1}^{n} \mu_i t + \frac{\sum_{i=1}^{n} \sigma_i^2 t^2}{2}}$$

By uniqueness MGF, $X_1 + X_2 + ... + X_n$ follows normal random variable with parameter $\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2 \right)$.

This proves the property.

b) Moment generating function of Normal distribution

$$= M_X(t) = E\left[e^{tx}\right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \dfrac{x-\mu}{\sigma}$ then $dz = \sigma dx, \; -\infty < Z < \infty$

$$\therefore M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu) - \frac{z^2}{2}} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2}{2} - t\sigma z\right)} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2 + \left(\frac{\sigma^2 t^2}{2}\right)} dz \quad = \frac{e^{\mu t} e^{\frac{\sigma^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz$$

$\because$ the total area under normal curve is unity, we have $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t\sigma)^2} dz = 1$

Hence $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ $\therefore$ For standard normal variable $N(0,1)$

$$M_X(t) = e^{\frac{t^2}{2}}$$

Problem 29. (a). The average percentage of marks of candidates in an examination is 45 will a standard deviation of 10 the minimum for a pass is 50%.If 1000 candidates appear for the examination, how many can be expected marks. If it is required, that double that number should pass, what should be the average percentage of marks?

(b). Given that X is normally distribution with mean 10 and probability $P[X > 12] = 0.1587$. What is the probability that X will fall in the interval $(9,11)$.

Solution:

a) Let X be marks of the candidates

Then $X \sim N(42,10^2)$

Let $z = \frac{X-42}{10}$

$P[X > 50] = P[Z > 0.8]$

$= 0.5 - P[0 < z < 0.8] \quad = 0.5 - 0.2881 = 0.2119$

Since 1000 students write the test, nearly 212 students would pass the examination.

If double that number should pass, then the no of passes should be 424.

We have to find z_1, such that $P[Z > z_1] = 0.424$

$\therefore P[0 < z < z_1] = 0.5 - 0.424 = 0.076$

From tables, $z = 0.19$

$\therefore z_1 = \frac{50 - x_1}{10} \Rightarrow x_1 = 50 - 10z_1 \quad = 50 - 1.9 = 48.1$

The average mark should be 48 nearly.

b) Given X is normally distributed with mean $\mu = 10$.

Let $z = \frac{x-\mu}{\sigma}$ be the standard normal variate.

For $X = 12, z = \frac{12-10}{\sigma} \Rightarrow z = \frac{2}{\sigma}$

Put $z_1 = \frac{2}{\sigma}$

Then $P[X > 12] = 0.1587$

$$P[Z > Z_1] = 0.1587$$
$$\therefore 0.5 - p[0 < z < z_1] = 0.1587$$
$$\Rightarrow P[0 < z < z_1] = 0.3413$$
From area table $P[0 < z < 1] = 0.3413$
$$\therefore Z_1 = 1 \Rightarrow \frac{2}{\sigma} = 1$$
To find $P[9 < x < 11]$

For $X = 9,\ z = -\frac{1}{2}$ and $X = 11,\ z = \frac{1}{2}$

$\therefore P[9 < X < 11] = P[-0.5 < z < 0.5] \ = 2P[0 < z < 0.5] \ = 2 \times 0.1915 = 0.3830$

Problem. (a). In a normal distribution,31 % of the items are under 45 and 8% are over 64.Find the mean and standard deviation of the distribution.

(b). For a certain distribution the first moment about 10 is 40 and that the 4th moment about 50 is 48, what are the parameters of the distribution.

Solution:

a) Let μ be the mean and σ be the standard deviation.

Then $P[X \le 45] = 0.31$ and $P[X \ge 64] = 0.08$

When $X = 45$, $Z = \frac{45 - \mu}{\sigma} = -z_1$

$\therefore z_1$ is the value of z corresponding to the area $\int_0^{z_1} \phi(z)dz = 0.19$

$\therefore z_1 = 0.495$

$45 - \mu = -0.495\sigma$ ---(1)

When $X = 64$, $Z = \frac{64 - \mu}{\sigma} = z_2$

$\therefore z_2$ is the value of z corresponding to the area $\int_0^{z_2} \phi(z)dz = 0.42$

$\therefore z_2 = 1.405$

$64 - \mu = 1.405\sigma$ ---(2)

Solving (1) & (2) We get $\mu = 10$ (approx) & $\sigma = 50$ (approx)

b) Let μ be mean and σ^2 the variance then $\mu_1' = 40$ about A=10
$$\therefore \text{Mean } A + \mu_1' = 10 + 40 \text{ 10+40}$$
$$\Rightarrow \mu = 50$$
Also $\mu_4 = 48 \Rightarrow 3\sigma^4 = 48 \Rightarrow \sigma^2 = 4$

$\therefore$ The parameters are Mean $= \mu = 50$ and S.D $= \sigma = 2$.