Seat No.: _____   Enrolment No._____

GUJARAT TECHNOLOGICAL UNIVERSITY
B.E. Sem-I   Examination January 2010

Subject code: 110008   **Subject Name: Mathematics – I**
Date: 11 / 01 /2010   **Time: 11.00 am – 02.00 pm**
                                                   Total Marks: 70

Instructions:
1. **Attempt all questions.**
2. **Make suitable assumptions wherever necessary.**
3. **Figures to the right indicate full marks.**

Q1. **(a)** **(i)** Find the value of k so that the function given below is continuous at a given point $x=2$.   **02**

$$f(x) = \begin{cases} \dfrac{2^{x+2}-16}{4^x-16}, & x \neq 2 \\ k & , & x = 2 \end{cases}$$

(ii) State Sandwich theorem and using it find $\displaystyle\lim_{x\to 0} g(x)$ if $3 - x^3 \leq g(x) \leq 3\sec x$ **02**

for all x.

(b) **(i)** If $f(x)$ and $g(x)$ are continuous functions for $0 \leq x \leq 1$, could   **02**
$f(x)/g(x)$ possibly be discontinuous at a point in the interval $[0,1]$? Give reasons for your answer.

(ii) If $f:(a,b) \to \Re$ is differentiable at $c \in (a,b)$, then show that   **02**

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h}$$ exists and equals $f'(c)$. Is the converse true?

(c) **(i)** Using Mean Value Theorem, Prove $0 < \dfrac{1}{x}\log\left(\dfrac{e^x-1}{x}\right) < 1$, *for* $x > 0$.   **03**

(ii) For what values of a, m and b does the function   **03**

$$f(x) = \begin{cases} 3 & , & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b & , & 1 \leq x \leq 2 \end{cases}$$

satisfy the hypothesis of the Mean Value Theorem on the interval $[0,2]$.

Q2. **(a)** **(i)** Find the area of the region between the *x-axis* and the graph of   **02**
$f(x) = x^3 - x^2 - 2x, \ -1 \leq x \leq 2$.

(ii) Using Fundamental Theorem of Calculus find dy/dx if $y = \displaystyle\int_1^{x^2} \cos t\, dt$.   **02**

(iii) Evaluate the integral $\displaystyle\int_0^\infty \dfrac{dx}{x^2+1}$.   **02**

(b) **(i)** Find the absolute maximum and minimum values of the function on the given interval $f(t) = |t-5|, \ 4 \leq t \leq 7$.   **03**

(ii) Find the Taylor's series expansion of $f(x) = x^3 - 2x + 4, \ a = 2$.   **02**

1

(iii) The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$. **02**
Show that the value of c in the conclusion of the Mean Value Theorem for
$f(x) = \frac{1}{x}$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

OR

(b) **(i)** Test the convergence or divergence of the following series **(ANY TWO)** **04**

a. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n - 1}{3^n}$ b. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{-n}}$ c. $\displaystyle\sum_{n=0}^{\infty} n!(x-4)^n$

(ii) Using Riemann Sum show that $\displaystyle\int_a^b x \ dx = \frac{1}{2}(b^2 - a^2)$ **03**

Q3. **(a)** Suppose that $w = f(x, y)$, $x = g(r, s)$ and $y = h(r, s)$ then write the chain rule **05**
for $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$. Also evaluate $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ in terms of r and s if
$w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$.

(b) **(i)** Show that $\displaystyle\int_{-\infty}^{\infty} f(x) \ dx$ may not equal to $\displaystyle\lim_{b \to \infty} \int_{-b}^{b} f(x) \ dx$. **02**

(ii) If $u = \tan^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u$ **03**

(c) Find the length of the curve $y = \displaystyle\int_0^x \sqrt{\cos 2t} \ dt$ from $x = 0$ to $x = \frac{\pi}{4}$. **04**

OR

Q3. **(a)** Let $w = f(x, y, z)$ be a function of three independent variables, write the **05**
formal definition of the partial derivative for $\frac{\partial f}{\partial z}$ at (x_0, y_0, z_0). Using this
definition find $\frac{\partial f}{\partial z}$ at $(1, 2, 3)$ for $f(x, y, z) = x^2 y z^2$.

(b) **(i)** Show that **03**

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except at the origin.

(ii) Find $\frac{dw}{dt}$ if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$. **02**

(c) Find the volume of the solid generated by revolving the region bounded by **04**
$y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about the line $y = 2$.

Q4. **(a)** **(i)** Evaluate the integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) dx \, dy$ **03**

(ii) Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and **04**
above the triangle enclosed by the lines $y = x, x = 0$ and $x + y = 2$ in the $xy-$
plane.

2

(b) Find the equations for tangent plane and normal line at the point $(1,1,1)$ on the surface $x^2 + y^2 + z^2 = 3$.   **03**

(c) Find the area of the region that lies inside the cardioid $r = 1 + \cos \vartheta$ and outside the circle $r = 1$.   **04**

OR

Q4. **(a)** Evaluate $\int\limits_{0}^{1} \int\limits_{0}^{\sqrt{z}} \int\limits_{0}^{2\pi} \left(r^2 \cos^2 \vartheta + z^2 \right) r \, d\vartheta \, dr \, dz$.   **04**

(b) **(i)** Integrate $f(x,y) = \dfrac{\ln(x^2 + y^2)}{\sqrt{x^2 + y^2}}$ over the region $1 \le x^2 + y^2 \le e$ by changing to polar coordinates   **04**

     (ii) Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$.   **02**

(c) Find the volume of the prism whose base is the triangle in xy - plane bounded by the x – axis and the line $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x,y) = 3 - x - y..$ **04**

Q5. **(a)** Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path $C = C_1 \cup C_2$ from $(0,0,0)\ to\ (1,1,1)$ with   **05**

$C_1 : r(t) = t\,\vec{i} + t^2\,\vec{j},\ 0 \le t \le 1$

$C_2 : r(t) = \vec{i} + \vec{j} + t\,\vec{k},\ 0 \le t \le 1$

(b) State Green's theorem and also evaluate the integral $\oint\limits_{C} (6y + x)dx + (y + 2x)dy$ where $C :$ The circle $(x - 2)^2 + (y - 3)^2 = 4$.   **05**

(c) Trace the curve $r^2 = a^2 \cos 2\vartheta.$ **04**

OR

Q5. **(a)** Use Green's theorem to evaluate the integral $\oint\limits_{C} \left(y^2 dx + x^2 dy \right)$ where $C :$ The triangle bounded by $x = 0$, $x + y = 1$, $y = 0$.   **05**

(b) Find the flux of $F = yz\,\vec{j} + z^2\,\vec{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1$, $z \ge 0$, by the planes $x = 0$ and $x = 1$.   **05**

(c) Use Stoke's theorem to evaluate $\int_C F \cdot dr$ if   **04**

$F = (x + y)\,\vec{i} + (2x - z)\,\vec{j} + (y + z)\,\vec{k}$ and C is the boundary of the triangle $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$.

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