V-Ex-I-09-C-5 M.A. / M.Sc. (Mathematics) (Part - I)

Con. 1417-09. Mathematics : Paper - II Analysis - I **MS-6978**

17/4/09

Internal (Scheme B)] (2 Hours) **[Total Marks : 40**

External (Scheme A)] (3 Hours) **[Total Marks : 100**

N.B. : (1) Write on the top of your answer book the Scheme underwhich you are appearing.

(2) Students of **Scheme B** answer **three** questions ; Students of **Scheme A** answer **five** questions.

(3) **All** questions carry **equal** marks.

1. (a) Show that every bounded monotone sequence of real numbers is convergent.

   (b) Let $\{ a_n \}$ be a sequence in $\mathbb{R}$ such that $\lim\inf a_n = \lim\sup a_n$. Show that $\{ a_n \}$ either converges or diverges.

   (c) Find limit inferior and limit superior of the sequence

$$\left\{\left(1+\frac{1}{n}\right)\cos n\pi : n \in \mathbb{N}\right\}.$$

2. (a) State and prove Leibniz theorem on the convergence of an alternating series.

   (b) Show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ is conditionally convergent.

   (c) Prove that $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^2}$ is convergent by quoting explicitly the results used.

3. (a) Find the derivative of $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1x_2$ at the point $(1, 1)$.

   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by—

$$f(x_1, x_2) = \frac{x_1 x_2}{x_1^2 + x_2^2} \text{ if } (x_1, x_2) \neq (0, 0)$$

$$f(0, 0) = 0.$$

   Verify if f is differentiable at (0, 0) and if the partial derivatives of f at (0, 0) exist or not.

   (c) Let E be an open set in $\mathbb{R}^n$ and f be a real valued function on E such that all partial derivatives $\frac{\partial f}{\partial x_j}$, $1 \leq k \leq n$ are continuous on E. Prove that f is differentiable on E and that its derivative at a point $a \in E$ corresponds to the linear operator L, given by—

$$L(t_1, t_2, \ldots, t_n) = \sum_{k=1}^{n} t_k \frac{\partial f}{\partial x_k}(a),$$

   for $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$.

[TURN OVER

4. (a) Prove that any increasing function $f : \mathbb{R} \rightarrow \mathbb{R}$ has atmost countable number of discontinuities and that they are of first kind.

   (b) If $f : [a, b] \mathbb{R}$ is differentiable on (a, b) and has bounded derivative, prove that f is of bounded variation on [a, b].

5. (a) Show that a continuous function f on [a, b] is Riemann-integrable.

   (b) If f is also non-negative in [a, b] and its integral over [a, b] varishes, show that f = 0 in [a, b].

6. (a) If f is continuous on [a, b], prove that $F'(x) = f(x)$ for all $x \in [a, b]$, where $F(x) = \int_a^x f(t)dt.$

   (b) If f and g are two Riemann integrable functions on [a, b] and $h : [a, b] \times [a, b] : \rightarrow \mathbb{R}$ is defined by $h(x, y) = f(x) \cdot g(y)$, is h Riemann integrable ? Justify your answer.

7. (a) State and prove Taylor's theorem for n–times continuously differentiable function of two variables.

   (b) Find and classify the extreme values of the following functions :—

   (i) $f(x) = x^2 + y^2 + xy + x + y$ and (ii) $f(x, y) = y^2 - x^3$

8. (a) State only Inverse Function Theorem. Use it to prove that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is one-one, continuously differentiable and has invertible Jacobian matrix at every point then f is an open mapping and $f^{-1} : f(\mathbb{R}^2) \rightarrow \mathbb{R}^2$ is differentiable.

   (b) Construct at C^{∞} real valued function on R^2 such that it is 1 on $\{ (x, y) : x^2 + y^2 \leq 1 \}$ and it is zero on $\mathbb{R}^2 \backslash \{(x, y) : x^2 + y^2 > 2 \}$.

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Analysis I

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Con. 1041–08. **NB–4228**

Internal Scheme B] **(2 Hours)** **[Total Marks : 40**

External Scheme A] **(3 Hours)** **[Total Marks : 100**

N.B. (1) Write on the top of your answer book the scheme under which you are appearing.
(2) Scheme-B students answer any three questions selecting atleast one from each section.
(3) Scheme-A students answer any five questions selecting atleast two from each section.
(4) All questions carry equal marks.
(5) Answers to both the sections are to be written in the same answer book.

SECTION I

1(a) For a non-empty bounded subset E of $\mathbb{R}$, define the least upper bound lub(E) and the greatest lower bound glb(E). Prove: If E and F are non-empty, bounded subset of $\mathbb{R}$ then (i) $\text{lub}(E+F) = \text{lub}(E) + \text{lub}(F)$ and (ii) $\text{lub}(4E) = 4\text{lub}(E)$.

1(b) Prove: If x and y are any two real numbers with $x > 0$, then there exists a natural number n such that $n \cdot x > y$.

2(a) Let $(x_n : n \in \mathbb{N})$ and $(y_n : n \in \mathbb{N})$ be sequences of real numbers converging to the real numbers l and m respectively. Prove that the sequence $(x_n \cdot y_n : n \in \mathbb{N})$ converges to $l \cdot m$.

2(b) Let the sequence $(x_n : n \in \mathbb{N})$ be given by $x_1 = \sqrt{2}$ and $x_n = \sqrt{2 + \sqrt{x_{n-1}}}$ for $n > 1$. Prove that the sequence $(x_n : n \in \mathbb{N})$ converges and find the limit.

3(a) A is a non-empty, closed subset of $\mathbb{R}$. Define $f : \mathbb{R} \to [0, \infty)$ by $f(x) = \text{glb}\{|x - a| \; : \; a \in A\}$. Prove: (i) f is uniformly continuous on $\mathbb{R}$ and (ii) $f(x) = 0$ if and only if $x \in A$.

3(b) Let A be a non-empty, closed subset and U an open subset of $\mathbb{R}$ such that $A \subset U$. Prove that there exists a continuous function $f : \mathbb{R} \to [0,1]$ such that $f \equiv 1$ on A and $f \equiv 0$ on $\mathbb{R} \setminus U$.

4(a) Let A be a non-empty subset of $\mathbb{R}$. Let $(f_n : A \to \mathbb{R} \; : \; n \in \mathbb{N})$ be a sequence of functions converging to a function $f : A \to \mathbb{R}$ the convergence being uniform on A. Suppose the limits $\lim_{x\to a} f_n(x)$ $(n \in \mathbb{N})$ and $\lim_{x\to a} f(x)$ exist for all a in A. Prove: the double limit $\lim_{n\to\infty} \lim_{x\to a} f_n(x)$ exists and is equal to $\lim_{x\to a} f(x)$ for all a in A.

4(b) If in question 4(a) above all functions f_n are continuous then prove that f is also continuous on A.

SECTION II

5(a) Let Ω be an open subset of $\mathbb{R}^n$ and $p \in \Omega$. Define differentiability of $f : \Omega \to \mathbb{R}^m$ at p. Suppose $f : \Omega \to \mathbb{R}^m$ is differentiable at p and $g : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $f(p)$. Prove that $g \circ f$ is differentiable at p and obtain the expression for the derivative of $g \circ f$ in terms of the derivatives of f and g.

5(b) Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear form. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = B(x, x)$ for all $x \in \mathbb{R}^n$. Prove that f is differentiable at every point $p \in \mathbb{R}^n$ and find an expression for $Df(p)$.

6(a) State without proof the inverse function theorem for a continuously differentiable function $f : \Omega \to \mathbb{R}^n$, Ω being an open subset of $\mathbb{R}^n$ and p, a point in Ω with the property that the derivative $Df(p)$ of f at p is non-singular.

6(b) State the implicit function theorem for a continuously differentiable map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and deduce the same from the inverse function theorem.

7(a) Define- giving all the relevant details- the Riemann integrability and the Riemann integral of a bounded function $f : R \to \mathbb{R}$, R being a closed, bounded rectangle in $\mathbb{R}^n$.

7(b) Explain how a continuous and compactly supported function $g : \mathbb{R}^n \to \mathbb{R}$ can be integrated in the sense of Riemann.

8(a) Prove: If $f, g : \mathbb{R} \to \mathbb{R}$ on a closed,bounded rectangle R in $\mathbb{R}^n$ are Riemann integrable, then $f + g$ is also Riemann integrable.

8(b) Prove: If $f : \mathbb{R} \to \mathbb{R}$ on a closed,bounded rectangle R in $\mathbb{R}^n$ is Riemann integrable then $|f|$ is also Riemann integrable.

Analysis I

Con. 3813–07. **KD–2229**

Internal (Scheme B) (2 Hours) **[Total Marks : 40**

External (Scheme A) (3 Hours) **[Total Marks : 100**

N.B. (1) Scheme-B students answer any three questions selecting atleast one from each section.
(2) Scheme-A students answer any five questions selecting atleast two from each section.
(3) All questions carry equal marks.
(4) Write on the top of your answer book the scheme under which you are appearing.
(5) Answers to both the sections are to be written in the same answer book.

SECTION I

1(a) Show that for any positive real number x and every natural number n there is a unique positive real number y such that $y^n = x$.

1(b) If m, n, p, q are integers, $n > 0, q > 0$ and $r = \frac{m}{n} = \frac{p}{q}$, prove that if $b > 0$ then $(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}$. Deduce that it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$

2(a) Define a subsequential limit of a sequence $\{p_n\}$ in $\mathbb{R}$ and show that the set of all subsequential limits of $\{p_n\}$ is a closed subset of $\mathbb{R}$.

2(b) Give an example of a sequence which has 11, 12, 13, 14, 15 as subsequential limits and no more.

3(a) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that f maps an interval onto an interval.

3(b) Let $f : [0,1] \to [0,1]$ be continuous. Show that there is $x \in [0,1]$ such that $f(x) = x$.

4(a) Determine the differentiability of the function if $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

4(b)) $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $D_{12}f(x,y)$ and $D_{21}f(x,y)$ exist at all points. If the functions $D_{12}f$ and $D_{21}f$ are continous, prove that $D_{12}f(x,y) = D_{21}f(x,y)\ \forall (x,y) \in \mathbb{R}^2$.

SECTION II

5(a) Let S be an open subset of $\mathbb{R}^n$. Let $f = (f_1, f_2, \cdots, f_n) : S \to \mathbb{R}^n$ be such that $D_j f_i$ are all continuous on S and that $J_f(a) \neq 0$ for some $a \in S$. Prove the following part of the Inverse Function Theorem: f is one-one on some neighborhood of a .

5(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x,y) = (e^x \cos y, e^x \sin y)$. Is f one-one? Is f onto $\mathbb{R}^2$? Justify.

6(a) State and prove second derivative test for local maxima and minima for function of two variables. State only it's analogue for function of $n-$variables.

6(b) Test the following functions for absolute maxima and minima.

(i) $f(x,y) = x^4 + y^4 - 2x^2 + 8y^2 + 4$ (ii) $f(x,y) = x^2 + xy + 3x + 2y$ (iii) $f(x,y) = 1 - x^2y^2$

7(a) Construct a C^∞-function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = 1$ if $\|x\| < 1$ and $f(x) = 0$ if $\|x\| \geq 2$.

7(b) When is a function $f : [0,1] \to \mathbb{R}$ is said to be of bounded variation. Show by means of an example that a continuous function need not be of bounded variation.

8(a) Prove that every continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$ is Riemann integrable.

8(b) Find the volume of the closed unit ball in $\mathbb{R}^3$.

Con/1215-07. SQ-4976

Internal(Scheme B)	(Time 2 Hours)	[Marks 40
External(Scheme A)	(Time 3 hours)	[Marks 100

N.B. (1) Scheme-B students answer any three questions selecting atleast one from each section.
(2) Scheme-A students answer any five questions selecting atleast two from each section.
(3) All questions carry equal marks.
(4) Write on the top of your answer book the scheme under which you are appearing.
(5) Answers to both the sections are to be written in the same answer book.

SECTION I

1(a) Define the term 'ordered field' and show that the field $\mathbb{C}$ of complex number is not an ordered field.

1(b) State the archimedean property of the field $\mathbb{R}$ of real numbers and use it to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

2 Let $\{s_n\}$ be a given sequence of real numbers and $s^* = \limsup s_n$.

2(a) Show that there is a subsequence $\{s_{n_k}\}$ of $\{s_n\}$ such that $s_{n_k} \to s^*$.

2(b) Show that if $x > s^*$ then there is an integer N such that $s_n < x$ for all $n > N$.

3(a) Show that a function $f = (f_1, f_2, \cdots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if each of $f_1, f_2, \cdots, f_m$ is continuous.

3(b) Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Show that for any $x \in (a, b)$ we have $f(x-) \leq f(x) \leq f(x+)$ and use it to prove that f has atmost countably many discontinuities.

4(a) Show by means of an example that a function $f : \mathbb{R}^n \to \mathbb{R}$ may have all its partial derivatives without it being differentiable. Justfy your statements.

4(b) Prove that a real valued function defined on $\mathbb{R}^n$ such that all its partial derivatives exist at all points and define continuous functions is differentiable.

SECTION II

6(a) State only Taylor's Theorem for real valued function of a real variable with any one form of remainder. Use it to state and prove its anlogue for function of $n-$ variables.

6(b) Prove that any triangle of maximal area inscribed in a circle of radius r is an equilateral triangle.

7(a) State only inverse function Theorem. If a C^1-function $f : \mathbb{R}^n \to \mathbb{R}^n$ is one-one and $Df(x)$ is non-singular for every $x \in \mathbb{R}^n$ then prove that $G = f(\mathbb{R}^n)$ is open in $\mathbb{R}^n$ and that f is a diffeomorphism from $\mathbb{R}^n$ onto G.

7(b) A continuous functon $f : [0, 1] \to \mathbb{R}$ is differentiable on $(0, 1)$ and has bounded derivative. Prove that f is of bounded variation on $[0, 1]$.

8(a) Let $I = [0, 1]$ and $R = I \times I$. When is a bounded function $h : R \to \mathbb{R}$ said to be Riemann integrable on R? Suppose $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are two Riemann integrable functions and $h(x, y) = f(x) \cdot g(y) \; \forall \; (x, y) \in R$. Is h Riemann integrable on R? Justify your answer.

8(b) Evaluate using spherical polar coordinates $\int_E f(x, y, z) d(x, y, z)$ where

$$E = \{(x, y, z) \in \mathbb{R}^3 / 1 \leq x^2 + y^2 + z^2 \leq 4\} \quad \text{and} \quad f(x, y, z) = xy(x^2 + y^2 + z^2)$$

V-1-09-B-Scan-55

mathematics : Paper-I - Algebra - I

Con. 1644-09. **MS-8184**

15/4/09

External [Scheme A] **(3 Hours)** **[Total Marks : 100**

Internal [Scheme B] **(2 Hours)** **[Total Marks : 40**

N.B.: (i) **External (Scheme A)** Candidates should attempt any **five (5)** questions.

(ii) **Internal (Scheme B)** candidates should attempt any **three (3)** questions.

(iii) **All** questions carry **equal** marks.

(iv) **Mark clearly the scheme under which you are appearing for the examination.**

1. (a) Let G be a finite cyclic group of order n. Show that the number of generators of G is the number of positive integers less than n and are prime to n.
   (b) Show that there is no nontrivial homomorphism from $\mathbb{Z}_{14}$ to $\mathbb{Z}_{15}$.

2. (a) Define the terms Euclidean domain, 'Principal Ideal Domain' (PID) and 'Unique Factorisation Domain' (UFD). Show that every PID is a UFD.
   (b) Show that $\mathbb{Z}[\sqrt{-2}]$ is Euclidean.

3. (a) Show that the center of S_n, the permutation group on $\{1, 2, ..., n\}$ is trivial for all $n \geq 3$.
   (b) Show that every group of order 4 is abelian.

4. (a) State and prove Eisenstein's criterion for irreducibility of a polynomial.
   (b) Show that $\mathbf{R}[X]/(X^2 + 1)$ is a field.

5. (a) Let V and W be two vector spaces over a field k. Assume that $Dim(V) = n$ and $Dim(W) = m$. Show that there is a 1-1 correspondence between the set of linear transformations from V to W and the set of $(m \times n)$ matrices over k.
   (b) In k^3, define T by $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2, -2x_2 + 2x_3)$. Show that T is a linear transformation. Calculate kernel of T.

6. (a) Define the terms *Bilinear form; Quadratic form.* Assume that the characteristic of the field is zero. Show that given a symmetric bilinear form f on a finite dimensional vector space V, there exists an ordered basis for V in which matrix of f is a diagonal matrix.
   (b) Let V be the vector space $\mathbf{R}$ over itself. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$. Define $f(\alpha, \beta) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$. Find a basis for V in which the matrix of f is diagonal.

7. (a) Define the Determinant function on $M_n(k)$, where k is any field. Show that Determinant function exists.
   (b) Let A be a 5×5 matrix with characteristic polynomial $f(x) = (x - 2)^3(x + 7)^2$ and minimal polynomial same as characteristic polynomial. What is the Jordan canonical form of A?

8. (a) Let T be a linear operator on a finite dimensional vector space V over a field k. Show that T is triangulable if and only if the minimal polynomial of T is a product of distinct polynomials over k.
   (b) Let T be a linear operator on a finite dimensional vector space V. Let W be a subspace of V with $T(W) \subset W$. Show that the minimal polynomial of $T|W$ divides the minimal polynomial of T.

M.A. MSC (Part - I) (Paper - I) Algebra - I

Mathematics.

Con. 4306–08. SM–4040

External [Scheme A] (3 Hours) [Total Marks : 100

External [Scheme B] (2 Hours) [Total Marks : 40

Instructions to the candidates.

(i) State clearly on the **Top Left Hand Corner** of the answer sheet, the scheme under which you are appearing for the examination.

(ii) Candidates appearing for the **Internal Scheme (Scheme B)** should attempt **three** questions.

(iii) Candidates appearing for the **External Scheme (Scheme A)** should attempt **five** questions.

(iv) All questions carry **equal** marks.

(v) Through out the paper, R denotes a commutative ring with identity and K denotes a field, unless otherwise stated. All rings considered are commutative rings with identity.

(1) (a) Define *conjugacy classes* and *centre* of a group G. Let G be a finite group. Let $Z(G)$ denote its centre and C_g denote the conjugacy class of $g \in G$. Prove the class equation:

$$|G| = |Z(G)| + \sum_{g \in G, |C_g| > 1} |C_g|.$$

(b) Prove that $|Z(G)| > 1$, if G is a group of order p^n, p a prime and $n \in \mathbb{N}$.

(2) (a) Let G be an abelian group. If $a, b \in G$ are elements of order m, n respectively, prove that there exists an element in G whose order is the least common multiple of m and n.

(b) Let $G = (\mathbb{Z}/n\mathbb{Z})^*$ denote the group of invertible elements of $\mathbb{Z}/n\mathbb{Z}$. Prove that $(\mathbb{Z}/n\mathbb{Z})^*$ is cyclic if n is a prime. Determine G if $n = 55$.

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(3) (a) If I, J are ideals of R such that $I + J = R$, prove that $I \cap J = IJ$ and that $R/IJ \cong R/I \times R/J$.

(b) Give an example of a ring R and ideals I, J such that $I \cap J \neq IJ$.

(4) (a) Prove that the ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean domain.

(b) Prove that $\mathbb{Z}[\sqrt{-7}]$ is not a Euclidean domain.

(5) (a) Let V be a vector space over a field K. If V has a finite basis, prove that every basis of V is finite and in that case prove that any two bases of V have the same number of elements.

(b) Let $f(X) \in K[X]$ be a polynomial of degree n. Prove that $K[X]/\langle f(X)\rangle$ is a vector space over K of dimension n, where $\langle f(X)\rangle$, denotes the ideal of $K[X]$, generated by $f(X)$.

(6) (a) If A is a square matrix over K, prove that there exists a monic polynomial $f(X) \in K[X]$ such that $f(A) = 0$.

(b) Find a monic polynomial $f(X) \in \mathbb{Q}[X]$ such that $f(A) = 0$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(7) (a) If A is a real symmetric matrix, prove that there exists an orthogonal matrix P such that PAP^{-1} is diagonal.

(b) Prove that the eigen values of a real symmetric matrix are real.

(8) (a) State and prove Silvester's law of inertia.

(b) Determine the signature of the following real symmetric matrix.

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Algebra I

Con/1209-07. SQ-4973

External(Scheme A) (Time 3 hours) Marks 100
Internal(Scheme B) (Time 2 Hours) Marks 40

N.B. (1) Scheme-B students answer any three questions.
(2) Scheme-A students answer any five questions.
(3) All questions carry equal marks.
(4) Write on the top of your answer book the scheme under which you are appearing.
(5) Answers to both the sections are to be written in the same answer book.
(6) Throughout the paper, R denotes a commutative ring with identity and F denotes a field, unless otherwise stated. All rings considered are commutative rings with identity.

Section I

1. (a) If p is a prime number, determine upto isomorphism all abelian groups of order p^3.
   (b) If the order of a finite abelian group G is divisible by 10 then show that G has a cyclic subgroup of order 10.

2. (a) Let G be a group, let $Z(G)$ denote the center of G and let $\mathrm{Inn}(G)$ denote the group of inner automorphisms of G. Show that $Z(G)$ is a normal subgroup of G and $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$.
   (b) Let A_4 denote the group of even permutations on the set $\{1, 2, 3, 4\}$. Show that A_4 does not have a subgroup of order 6.

3. (a) Show that $\mathbb{Z}[i]$, the ring of Gaussian integers is a euclidean domain.
   (b) In $\mathbb{Z}[\sqrt{-5}]$, show that 21 does not factor uniquely as a product of irreducibles.

4. (a) Let F be a field and let $p(X) \in F[X]$. Show that $< p(X) >$ is a maximal ideal in $F[X]$ if and only if $p(X)$ is irreducible over F.
   (b) Find all monic irreducible polynomials of degree 2 over $\mathbb{Z}/5\mathbb{Z}$

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Section II

5. (a) If V is a finite dimensional vector space over a field F, and V^* denotes the dual vector space of V, prove that $V \cong V^*$.

   (b) Let V be a finite dimensional vector space over a field F. For any subspace W of V let W^0 denote the annihilator of W. Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

6. (a) Let T be a linear operator on V, where V is as above. If M, N are matrices of T corresponding to two ordered bases of V, prove that the determinant of M equals the determinant of N.

   (b) Let V be the space of all $n \times 1$ column matrices over a field F. Show that every linear operator on V is left multiplication by a unique $n \times n$ matrix over F.

7. (a) Let T be a linear operator on a finite dimensional vector space V. Define characteristic and minimal polynomials of T and show that they have the same roots except for multiplicities.

   (b) Let T be a diagonalizable linear operator on $\mathbb{R}^5$ with set of characteristic values $\{1, 3, 7\}$. What will be the minimal polynomial of T? Justify your answer.

8. (a) For any linear operator T on a finite dimensional inner product space V with inner product $< \, , \, >$, show that there exists a unique linear operator T^* on V such that $< Tv, \overline{w} >=< v, T^*w >$ for all $v, w \in V$.

   (b) Let V be a finite dimensional complex inner product space. Let T be a linear operator on V. If T is unitary, prove that there is a basis of V with respect to which the matrix of T is diagonal.

Algebra - I

30 : 2ndH107

Con. 3798-07. KD-2205

External Scheme A] (3 Hours) [Total Marks : 100
External Scheme B] (2 Hours) [Total Marks : 40

N.B.(1) **Scheme B** students answer any **three** questions.
(2) **Scheme A** students answer any **five** questions.
(3) **All** questions carry **equal** marks.
(4) Write on the **top** of your answer-book the **scheme** under wich you are appearing.
(5) Answers to **both** the sections are to be written in the **same** answer-book.
(6) Throughout the paper, R denotes a commutative ring with identity and F denotes a field, unless otherwise stated. All rings considered are commutative rings with identity.

Section I

1. (a) Let G be a finite abelian group of order $p^e m$ where p is a prime that does not divide m. Show that G is the internal direct product of two subgroups H and K where $|H| = p^e$ and $|K| = m$.

   (b) Prove or disprove: Every abelian group of order 180 has a cyclic subgroup of order 18.

2. (a) Prove that the center of a group of order p^n, where p is a prime number and n is a natural number, is non trivial.

   (b) Let p be a prime number. Prove that a group of order p^2 is abelian.

3. (a) Show that $\mathbf{Z}[\omega]$, where ω is a primitive cube root of unity, is a Euclidean domain.

   (b) Let$R = \mathbf{Z}[\omega]$. Prove or disprove: $R[X]$ is a unique factorization domain.

4. (a) Let F be a field and let $p(X) \in F[X]$. Show that $< p(X) >$ is a maximal ideal in $F[X]$ if and only if $p(X)$ is irreducible over F.

   (b) Find all monic irreducible polynomials of degree 2 over $\mathbf{Z}/2\mathbf{Z}$.

[TURN OVER

Section II

5. (a) Let V be a vector space over a field F. If there exists an infinite subset of V which is linearly independent over F, prove that the diemnsion of V must be infinte.

   (b) If V is a finite dimensional vector space over a field F, and V^* denotes the dual vector space of V, prove that $V \cong V^*$.

6. (a) Let V be a vector space of dimension n over a field F. Fixing an ordered basis of V, prove that there is a one to one correspondence between linear operators on V and $n \times n$ matrices over F.

   (b) Let T be a linear operator on V, where V is as above. If M, N are matrices of T corresponding to two ordered bases of V, prove that determinant of M equals the determinant of N.

7. (a) Let V be a finite dimensional vector space over a field F and suppose characteristic of F is not equal to 2. If B is a non singular symmetric bilinear form on V, prove that there exists a basis of V with respect to which the matrix of B is diagonal.

   (b) Let $V = \mathbb{R}^2$. Let $B : V \times V \to \mathbb{R}$ be the symmetric bilinear form defined by $B((x_1, x_2), (y_1, y_2)) = x_1y_1 - x_2y_2$. Find the matrix of B with respect to the standard basis of V and determine its signature.

8. (a) Show that every complex $n \times n$ matrix is similar over $\mathbb{C}$ to an upper triangular matrix.

   (b) If A is a complex nilpotent matrix, prove that the eigen values of $I + A$ are all equal to 1.

M.A & M.Sc (Mathematics) (Part - I)

Mathematics : Paper - III - Topology

Con. 1692–09. **MS–8187**

For Internal (Scheme B)] (2 Hours) **[Total Marks : 40**

For External (Scheme A)] (3 Hours) **[Total Marks : 100**

20/4/09

N.B. : (1) Write on the **top** of your answer book the **Scheme** under which you are appearing.

(2) Students of **Scheme B** answer **three** questions with at least **one** question from **each section**; Students of **Scheme A** answer **five** questions with at least **two** questions from **each** section.

(3) **All** questions carry **equal** marks. Answers to **both** the sections are to be written in the **same** answer book.

Section I

1. (a) Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of finite sets. If J is finite, then show that the sets

$$\bigcup_{\alpha \in J} A_\alpha \quad \text{and} \quad \prod_{\alpha \in J} A_\alpha$$

are both finite sets.

(b) Show that for any non-empty set A, the cardinality of the power set of A is strictly greater than that of A.

2. (a) Define a basis and a subbasis for a topology on X. Give an example of a subbasis which is not a basis.

(b) Show that the countable collection $\mathbf{B} = \{ (a,b) : a, b \in Q \}$ is a basis for the standard topology on R, while the countable collection $\mathbf{B} = \{ (a,b) : a, b \in Q \}$ generates a topology different from the lower limit topology on R.

(c) Let X be a topological space. Let Λ denote the subset $\{ x \times x : x \in X \}$ of $X \times X$. Show that X is a Hausdorff space if and only if Λ is a closed subset of $X \times X$.

3. (a) State and prove the pasting lemma to prove continuity of a given function.

(b) Give an example of continuous bijection from one topological space to the other which is not a homeomorphism.

(c) Let $f : X \longrightarrow Y$ be a function where X is a metric space. Show that the function f is continuous iff for every convergent sequence $x_n \longrightarrow x$ in X the sequence $f(x_n)$ converges to f(x) in Y.

4. (a) Define a connected subspace of a topological space X. Show that if A is a connected subspace of X and if $A \subseteq B \subseteq \bar{A}$, then B is connected.

(b) Let $p \in X$ and let $\{ A_i : i \in I \}$ be a family of connected subsets of X such that $p \in A_i$ for every $i \in I$. Show that $\cup_{i \in I} A_i$ is connected.

(c) Show that product of two connected topological spaces is connected.

[TURN OVER

Section II

5. (a) Let Y be a subspace of X. Show that Y is compact iff every covering of Y by sets that are open in X has a finite subcollection covering Y.

   (b) Show that every closed subset of a compact space is compact.

   (c) Show that every locally compact Hausdorff space X which is not compact, has a one-point compactification Y such that Y is compact Hausdorff and $\bar{X}$ equals Y.

6. (a) Let X be the set R in the lower limit topology. Show that X is a Lindeloff space but $X \times X$ is not a Lindeloff space.

   (b) Show that a closed subspace of a normal space is normal.

7. (a) Let X be a metric space. Show that if every Cauchy sequence in X has a convergent subsequence, then X is complete.

   (b) Let X be a metric space. Show that X is compact iff X is a complete and totally bounded metric space.

   (c) Let C be the set of all continuous real valued functions on [0, 1] equipped with the sup metric. Let F be a subset of C. Show that if F is an equicontinuous family, then so is $\bar{F}$.

8. (a) Let $p : E \to B$ be a covering map. Let B be connected. Show that, if for some b_0, the set $p^{-1}(b_0)$ has k elements, then for every b, the set $p^{-1}(b)$ has k elements.

   (b) Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Show that any path $f : [0, 1] \to B$ beginning at b_0 has a unique lifting to a path $\bar{f}$ in E beginning at e_0.

Topology

Mathematics.

Con. 4352–08. **SM–4067**

Scheme A] (3 Hours) **[Total Marks : 100**

N.B. : Answer any **four** questions.

Scheme B] (2 Hours) **[Total Marks : 40**

N.B. : Answer any three questions.

Topology

1. (a) Give an example with details to show that a countable product of countable sets need not be countable.

   (b) Let $\mathcal{A} = \{f \mid f : \mathbb{N} \longrightarrow \{0,1\}\}$ be the collection of all maps from $\mathbb{N}$ to $\{0,1\}$. Construct an injective map from $\mathbb{R}$ into $\mathcal{A}$.

2. (a) State and prove the 'Pasting Lemma'.

   (b) Let $f, g : [0,1] \longrightarrow X$ be continuous maps into a topological space X such that $f(1) = g(0)$. Define $h : [0,1] \longrightarrow X$ by $h(s) = f(2s)$ for all $s \in [0, \frac{1}{2}]$ and $h(s) = g(2s-1)$ for all $s \in [\frac{1}{2}, 1]$. Verify that h is a continuous map.

3. (a) Let X be a topological space. Prove that X is a connected space if and only if every continuous map from X to the discrete space $\{0,1\}$ is a constant function.

   (b) Define a path connected space. Prove that any open, connected subset of $\mathbb{R}^n$ is path connected.

4. (a) Prove that any compact subset of a Hausdorff space X is closed in X.

   (b) State and prove the 'Tube Lemma'.

5. Let X, Y, Z be topological spaces. S^1 denotes the unit circle in $\mathbb{R}^2$ with center at $(0,0)$

   (a) Let $f : X \longrightarrow Y$ be a map. When do you say f is a 'quotient map'. Show that $g : \mathbb{R} \longrightarrow S^1$ defined by $g(x) = (\cos x, \sin x)$ $(x \in \mathbb{R})$ is a quotient map.

   (b) Let $\eta : X \longrightarrow Y$ be a quotient map. Suppose $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ be maps with f being continuous and $g \circ \eta = f$. Then show that g is a continuous map.

6. (a) Define the 'interior of a subset' in a space. Define a 'Baire space'. Prove that a compact, Hausdorff space is a Baire space.

   (b) Prove that $\mathbb{Q}$ can not be written as intersection of countably many open subsets of $\mathbb{R}$.

7. (a) Define the notion of 'path homotopy'. Let $\alpha : [0,1] \longrightarrow X$ be a path in a space X with $\alpha(1) = q$. Prove that $\alpha * c_q$ is path homotopic to α $(c_q(s) = q$ for all $s \in [0,1])$.

   (b) Prove that $f : S^1 \longrightarrow S^1$ defined by $f(z) = z^2$ is a covering map.

M.A. & M.Sc. (PART - I)
Mathematics : paper III - Topology
21st April 2008.

P4/RT-Ex-08-424

Con. 1335–08.

NB–5323

Scheme A] (3 Hours) **[Total Marks : 100**

N.B. : Answer any **four** questions.

Scheme B] (2 Hours) **[Total Marks : 40**

N.B. : Answer any **three** questions.

1. (a) Prove that a finite product of countable sets is countable.
   (b) Consider $\mathcal{A} = \{S \subset \mathbb{N} \mid S \text{ is an infinite set}\}$. Prove that $\mathcal{A}$ is an uncountable set.

2. (a) State and prove the 'Pasting Lemma'.
   (b) Consider the subsets A, B, C of $\mathbb{R}^2$ defined by
   $A = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}$
   $B = \{(x, y) \in \mathbb{R}^2 \mid (x-1/2)^2 + y^2 = 1/4\}$
   and $C = \{(x, y) \in \mathbb{R}^2 \mid (x+1/2)^2 + y^2 = 1/4\}$.
   Construct a continuous bijection from $A \cup B$ onto $A \cup C$.

3. (a) Give an example of a connected space which is not path connected. Justify.
   (b) Prove that $\mathbb{R}^n \setminus \{0\}$ $(n > 1)$ is connected.

4. (a) State and prove the 'Tube Lemma'.
   (b) Let X, Y be topological spaces. If Y is compact, then prove that the projection $\pi_1 : X \times Y \longrightarrow X$ is a closed map.

5. (a) Define the terms: Second Countable Space, Separable Space. Prove that a second countable space is separable.
   (b) Find a countable dense subset of the irrational numbers. Justify.

6. (a) Prove that there does not exist a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous only at rational numbers.
   (b) Prove that a locally compact, Hausdorff space is a Baire space.

7. (a) Let $p : E \longrightarrow B$ be a covering map. Assume B is path connected. For any $x, y \in B$, prove that $p^{-1}(x), p^{-1}(y)$ have same cardinality.
   (b) Define $c(t) = (\cos 2\pi t, \sin 2\pi t) \in S^1$ $(t \in [0, 1])$. Show that c is not path homotopic to a constant path in the space S^1.

M.Sc (Mathematics) - I - Topology (I)

KD–2214

3823–07.

rnal (Scheme B) (2 Hours) [Total Marks : 40

xternal (Scheme A) (3 Hours) [Total Marks : 100

N.B. (1) Scheme-B students answer any three questions selecting atleast one from each section.
(2) Scheme-A students answer any five questions selecting atleast two from each section.
(3) All questions carry equal marks.
(4) Write on the top of your answer book the scheme under which you are appearing.
(5) Answers to both the sections are to be written in the same answer book.

SECTION I

1(a) Define a countable set. A and B are countable sets. Prove that $A \times B$ is also countable.

1(b) Lat S be the set of sequences $s = (s_n)$ such that $s_n \in \{0, 1\}$ for all $n \in \mathbb{N}$. Show that S is not countable.

2(a) Let X be a topological space. Let $A \subset X$. Define ∂A-the boundry of A. Prove that ∂A is empty if and only if A is both open and closed.

2(b) Show that a separable metric space is second countable.

3(a) Let $f : X \to Y$ be a map of topological spaces. Prove that the fllowing two conditions are the equivalent: (i) $f^{-1}(F)$ is a closed set for every closed subset F of Y. (ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

3(b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x, |y|)$. Prove that f is a closed map.

4(a) Define a ' quotient map'. Suppose $\eta : X \to Y$ is a quotient map. Suppose $f : X \to Z$ and $g : Y \to Z$ are maps of topological spaces such that $g \circ \eta = f$. Show that f is continuous if and only if g is continuous.

4(b) If (X_1, τ_1) and (X_2, τ_2) are topological spaces, define product topology on $X = X_1 \times X_2$. Prove that the produc space X is separable if and only if both X_1 and X_2 are separable.

SECTION II

5(a) Prove that $[0, 1]$ is a compact subset of $\mathbb{R}$ provided with ususal topology.

5(b) Prove thayt every continuous map $f : X \to \mathbb{R}$ on a compact metric space X is uniformly continuous, bounded and attains the bounds.

6(a) Prove that every open subset of $\mathbb{R}$ can be written as a countable disjoint union of open intervals.

6(b) Find the connected components of $\{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \mid xy = 0\}$

7(a) Let α, β and γ be loops in the topological space X based at a point $a \in X$. Define $\alpha * \beta$. Prove that $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ are path homotopic.

7(b) Let $p : G \to B$ be a covering map. Assume that B is path connected. Show that there is a bijection from $p^{-1}(a)$ and $p^{-1}(b)$ for any two points a and b in B.

8(a) Consider the covering map $p : S^1 \to S^1$ defined by $p(z) = z^2$. Define $\gamma : [0, 1] \to S^1$ by $\gamma(s) = \cos(\pi s) + \sqrt{-1} \sin(\pi s)$. Find a path $\mu : [0, 1] \to S^1$ such that $\mu(0) = -1$ and $(p \circ \mu)(s) = \gamma(s)$ for all $s \in [0, 1]$.

8(b) Let X be a topological space and $A \subseteq X$. When do you say A is a retract of X? Prove that S^1 is a retract of $\mathbb{R}^2 \setminus \{(0, 0)\}$.