(b) If $f(z) = \int_{-A}^{A} F(t)\, e^{itz}\, dt$, where $0 < A < \infty$ and $F \in L^2(-A, A)$ prove that $|f(z)| \le Ce^{|A|Z}$ where $C = \int_{-A}^{A} | F(t) |\, dt$.

M.Phil. DEGREE EXAMINATION – JUNE 2006.

Mathematics

Paper III – FOURIER TRANSFORMS

Time : 3 hours Maximum marks : 75

Answer any FIVE questions.

Each question carries 15 marks.

1. (a) Prove that the interval (a, ∞) is measurable.

(b) Prove that the outer measure of a set is translation invariant.

2. (a) Prove that every Borel set is measurable.

(b) Let $< E_i >$ be a sequence of disjoint measurable sets and A any set. Then prove that $m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$.

3. (a) Prove that Lebesque measure is invariant under translation modulo 1.

(b) State and prove one version of Littlewood's third principle.

4. (a) Let $\{f_n\}$ be a sequence of measurable functions on R, and suppose that

(i) $0 \le f_1(x) \le f_2(x) \le .... < \infty$ for every $x \in R$,

(ii) $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in R$. Then prove that f is measurable, and $\int_R f_n \, d_\mu \to \int_R f d_\mu$ as $n \to \infty$.

(b) Prove that if f is a real function on R such that $\{x : f(x) \ge r\}$ is measurable for every rational number r, then f is measurable.

5. (a) Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e. on R such that $\sum_{n=1}^{\infty} \int_R |fn| \, d\mu < \infty$. Then prove that the series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for almost all x, $f \in L^1(\mu)$ and $\sum_{n=1}^{\infty} \int_R f_n \, d\mu = \int_R f \, d\mu$.

(b) Let $\{E_R\}$ be a sequence of measurable sets in R, such that $\sum_{R=1}^{\infty} \mu(E_R) < \infty$.

Then prove that almost all $x \in R$ 1 ie in at most finitely many of the sets E_R.

6. (a) Let (X, ζ, μ) and $(Y, \Im, \lambda)$ be σ-finite measure spaces. Suppose $Q \in \zeta \times \Im$. If

$\phi(x) = \lambda(Q_x), \Psi(y) = \mu(Q^y)$ for every $x \in X$ and $y \in Y$, then prove that ϕ is ζ-measurable, Ψ is $\Im$-measurable, and $\int_x \phi d\mu = \int_y \Psi d\lambda$.

(b) Prove that in Fubini's theorem the requirement that "σ-finiteness of measure spaces" cannot be dispensed with.

7. If f and g are in $L'(R')$ prove that the convolution h of f and g is also in $L'(R')$ and that $\|h\|_1 \le \|f\|_1 \|g\|_1$.

8. (a) If $f \in L'$ then prove that $\hat{f} \in C_0$.

(b) State and prove the Inversion theorem.

9. State and prove the Plancherel Theorem.

10. (a) Let $F \in L^2(0, \infty)$ and let

$$f(z) = \int_0^{\infty} F(t) e^{itz} \, dt, z \in \pi^+$$

where $\pi^+ = \{z = x + iy : y > 0\}$. Then prove that f is holomorphic in π^+ and its restriction to horizontal lines in π^+ form a bounded set in $L^2(-\infty, \infty)$.