7. (a) State and prove the spectral theorem for compact self-adjoint operators.

(b) Let A be a self-adjoint operator on H, $u \in c([m_A, m_A])$ and $U(A) \in BL(H)$. Prove that

$$\|U(A)\| = \sup\{|u(t)| : t \in S(A)\}.$$

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MPL–637	RMSD

M.Phil. DEGREE EXAMINATION – JUNE 2006

Mathematics

SPECTRAL THEORY

Time : 3 hours Maximum marks : 75

Answer any FIVE questions.

Each question carries 15 marks.

1. (a) Let $\{U_n : n = 1, 2, 3, \cdots\}$ be an orthonormal set in a Hilbert Space H. For a sequence (K_n) of scalars, prove that the following are equivalent.

(i) There exists $x \in H$ such that for $n = 1, 2, 3, \cdots \langle x, U_n \rangle = K_n$.

(ii) $\sum_{n=1}^{\infty} |K_n|^2 < \infty$ and

(iii) $\sum_{n=1}^{\infty} K_n U_n$ converges in H.

(b) Let $\{x_n : n = 1, 2, 3 \cdots\}$ be an orthogonal set in H. Then prove that $\sum_{n=1}^{\infty} x_n$ converges in H if and only if $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$.

2. (a) State and prove unique Hahn-Banach extension theorem.

(b) If (x_n) is a sequence in H such that $(\langle x_n, y \rangle)$ converges for every $y \in H$ then prove that there is a unique $x \in H$ such that (x_n) converges weakly to x in H.

3. (a) Let A be an operator on a Hilbert space H. Suppose there is an operator B on H such that $\langle A(x), y \rangle = \langle x, B(y) \rangle$ for all $x, y \in H$. Then prove that A is bounded and $B = A^*$.

(b) Let $A \in BL(H)$. Then prove that

(i) A is unitary if and only if $\|A(x)\| = \|x\|$ for all $x \in H$ and A is onto.

(ii) A is normal if and only if $\|A(x)\| = \|A^*(x)\|$ for all $x \in H$.

4. (a) Prove that $A \in BL(H)$ is invertible in $BL(H)$ iff A is bounded below and the range of A is dense in H.

(b) Let $K = C$. If $A \in BL(H)$, then prove that

(i) $\|A\| \leq 2 R_A$ and

(ii) $R_A{}^2 \leq (RA)^2$.

5. (a) Let $K = C$ and A be a normal operator. Prove that $\|A\| = R_A = r_A$.

(b) Let H be an n-dimensional Hilbert Space over K, and $A \in BL(H)$. Prove that there is an orthonormal basis for H consisting of eigen vectors of A iff A is normal in $K = C$ and iff A is self-adjoint in $K = R$.

6. (a) Let A be a non-zero compact self-adjoint operator on a real or complex Hilbert space. $\|A\|$ or $-\|A\|$ is an eigen value of A. Prove that there are only a finite number of linearly independent eigen vectors corresponding to this eigen value.

(b) Prove that a compact self-adjoint operator on a Hilbert space is positive iff all of its eigen values are non-negative.