PART A

- Find the correct answer and mark it on the answer sheet on the **top page**.
- A right answer gets **1 mark** and a wrong answer gets $-\mathbf{\frac{1}{3}}$ **mark**.

1. Class of all generalized inverses of a $m \times n$ real matrix A

   (a) is a non-empty vector space.
   (b) is a non-empty convex set.
   (c) can be empty.
   (d) is a non-empty finite set.

2. Consider a real valued function f such that for some x_0 in $\mathcal{R}$, $f(x_0) = e^{2/c}$, suppose $c \in (1, 2)$, the value of $f(x_0)$ can be

   (a) 4.
   (b) 2.
   (c) 1.
   (d) 0.

3. $\lim_{n\to\infty} \left(1 + \frac{1}{n}(1 + \frac{1}{n})^n\right)^n$ is

   (a) e.
   (b) e^2.
   (c) e^{e^2}.
   (d) e^e.

4. Let λ_1 and λ_2 be the characteristic roots (eigen values) of the matrix $A = \begin{pmatrix} -1 & 1/2 \\ 1/2 & -1 \end{pmatrix}$ then

   (a) both λ_1 and λ_2 are positive.
   (b) both λ_1 and λ_2 are negative.
   (c) one of λ_1 and λ_2 is positive and the other is negative.
   (d) one of λ_1 and λ_2 is zero and the other is positive.

5. Let R_i be the rank of i^{th} observation in a random sample of size N, $i = 1, 2, \ldots, N$, then $E(R_i)$ is

   (a) $\frac{N+1}{2}$.
   (b) $\frac{N}{N+1}$.
   (c) $\frac{N}{2}$.
   (d) $\frac{N+1}{N}$.

6. In a randomized block design (RBD) with 3 treatments, it is given that the ratio of degrees of freedom for treatments to that of total source of variation is 0.25. Hence the number of blocks is

   (a) 6.
   (b) 5.
   (c) 4.
   (d) 3.

7. Suppose X is $N(0,1)$ random variable and $Y = |X|$. Then the correlation coefficient between X and Y is

   (a) -1.
   (b) 0.5.
   (c) 0.
   (d) 1.

8. Suppose X and Y are two random variables with $E(Y|X) = X$. Then

   (a) $Var(Y) = Var(X)$.
   (b) $Var(Y|X) = Var(X)$.
   (c) $Cov(X,Y) = Var(X)$.
   (d) $Cov(X,Y) = Var(Y)$.

9. Let X_1, X_2 and X_3 be independent $U(0,1)$ random variable, $P(X_1 < X_2 < X_3)$ is

   (a) $\frac{1}{2}$.
   (b) $\frac{1}{3}$.
   (c) $\frac{1}{4}$.
   (d) $\frac{1}{6}$.

10. Let $X_1, \ldots, X_n$ be i.i.d from $U(\theta, \theta+1)$. Define $X_{(1)} = min\{X_1, \ldots, X_n\}$ and $X_{(n)} = max\{X_1, \ldots, X_n\}$, then

    (a) $X_{(1)}$ is sufficient for θ.
    (b) $\left(X_{(1)}, X_{(n)}\right)$ is sufficient for θ.
    (c) $X_{(n)}$ is sufficient for θ.
    (d) $X_{(n)}^2$ is sufficient for θ.

11. Let $X_1, X_2, \cdots, X_n$ be a random sample with common location and scale parameters μ and σ^2 respectively, then the statistic $\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2}$ is

    (a) location invariant.
    (b) scale invariant.
    (c) both location invariant and scale invariant.
    (d) none of the above.

12. Let X_1, X_2 be i.i.d $N(\theta, 1)$. Let $\varphi(X_1) = E_\theta(\overline{X}|X_1)$, then $\varphi(X_1)$ is

    (a) $\frac{1}{2}X_1 + \frac{1}{2}\theta$.
    (b) $X_1 + \theta$.
    (c) $\frac{1}{2}X_1 + \theta$.
    (d) $X_1 + \frac{1}{2}\theta$.

13. The proportion of households in a town with 0,1,2, 3 and more than 3 children is $1 - 10p, 2p, 4p, 3p$ and p respectively, $0 < p < \frac{1}{10}$. In a random sample of 10 households one household had no child, 3 had one child, 5 had 2 children and one had 3 children. The maximum likelihood estimate for the proportion of households with 2 children is

    (a) $\frac{2}{25}$.
    (b) $\frac{4}{25}$.
    (c) $\frac{9}{25}$.
    (d) $\frac{1}{2}$.

14. Let $X_1, ..., X_n$ be a random sample from the exponential distribution with mean λ. To test the hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda > \lambda_0$, the p value based on the test statistic using sample mean is p_0. X_n was wrongly observed to be 100 when the correct value was 120. Let p_1 be the p-value of the same procedure after making the required correction on X_n, then

    (a) $p_1 > p_0$.
    (b) $p_1 = p_0$.
    (c) $p_1 < p_0$.
    (d) p_1 will have no specific relation with p_0.

15. For a random variable X with parameter θ, if $L(.)$ and $U(.)$ satisfy $P_\theta\left(L(X) \leq \theta\right) = 1 - \alpha_1$ and $P_\theta\left(U(X) \geq \theta\right) = 1 - \alpha_2$ and $L(x) \leq U(x)$ for all x, then $P_\theta\left(L(X) \leq \theta \leq U(X)\right)$ is

    (a) $1 - \alpha_1 - \alpha_2$.
    (b) $\alpha_1 + \alpha_2 - 1$.
    (c) $\frac{\alpha_1+\alpha_2}{2}$.
    (d) $\frac{\alpha_1\alpha_2}{2}$.

16. Let P be a probability measure on the class of events on $\Omega = [0, \infty)$. Suppose $P\Big((a, b]\Big) = \int_a^b e^{-x}dx, \quad 0 \leq a \leq b \leq \infty$, further for any $E \subset \Omega$ and $z \in \mathcal{R}, E + z = \{x + z; x \in E\}$. Then $P\Big((2, 4] + 3\Big)$ is

    (a) less than $P\Big((2, 4]\Big)$.
    (b) greater than $P\Big((2, 4]\Big)$.
    (c) equal to $P\Big((2, 4]\Big)$.
    (d) cannot be determined.

17. The characteristic function $\varphi(t)$ of a random variable X is $\frac{1}{1+t^2}$, then $E(X)$

    (a) is 1.
    (b) is 0.
    (c) does not exist.
    (d) cannot be uniquely determined.

18. Suppose X_i, $i = 1, 2, ..., n$ are Bernoulli random variables on $\{-1, 1\}$ with mean $\frac{1}{2}$. Then the characteristic function of the random variable $Y = \sum_{i=1}^{n} X_i^2$ is

    (a) $\frac{\left(1+3e^{it}\right)}{4}$.
    (b) $\frac{\left(1+3e^{it}\right)^n}{4^n}$.
    (c) e^{nit}.
    (d) e^{3it}.

19. Let $\{X_n\}_1^\infty$ be a sequence of independent random variables with probability distributions as follows: $P(X_1 = 0) = P(X_1 = 2) = \frac{1}{2}; P(X_n = 1 - \sqrt{n}) = P(X_n = 1 + \sqrt{n}) = \frac{1}{n}$, $P(X_n = 1) = 1 - \frac{2}{n}, n = 2, 3, \cdots$. If $S_n = X_1 + ... + X_n$, then $\lim_{n \to \infty} P(S_n > n)$

   (a) is 0.
   (b) is 1.
   (c) is $\frac{1}{2}$.
   (d) does not exist.

20. A population of 60 units is split into 3 strata of equal sizes. The within stratum variances of the variable of interest Y are $\sigma^2, 4\sigma^2, 9\sigma^2$ for stratum $1, 2$ and 3 respectively. A stratified sample of 18 units is to be drawn, the optimal allocation of the sample from strata $1, 2, 3$ is respectively

   (a) 2, 8, 10.
   (b) 3, 6, 9.
   (c) 3, 7, 8.
   (d) 2, 5, 11.

21. If $\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\mathbf{0}, \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}\right)$ If $Y_1 = X_1 + X_2 + X_3$ and $Y_2 = X_1 + X_2 - 2X_3$, then

   (a) Y_1 and Y_2 are not independent.
   (b) Y_1 and Y_2 are uncorrelated but not independent.
   (c) Y_1 and Y_2 are correlated with correlation coefficient equal to 1/2.
   (d) Y_1 and Y_2 are independent.

22. The transition probability matrix of a Markov chain with state space $S = \{1, 2, 3\}$ is

$$\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2}. \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

   This Markov chain is

   (a) irreducible and hence recurrent.
   (b) not irreducible.
   (c) does not process a stationary distribution.
   (d) irreducible but not recurrent.

23. Suppose $E(Y|X) = 15X$, where $X \sim Beta(2, 1)$, then $E(Y)$ is

 (a) $\frac{15}{4}$.
 (b) 5.
 (c) $\frac{15}{2}$.
 (d) 10.

24. Let X be a standard normal random variable and $Y = max(0, X)$, then $E(Y)$ is

 (a) 0.
 (b) $\frac{1}{\sqrt{2\pi}}$.
 (c) $\frac{1}{\sqrt{4\pi}}$.
 (d) $\frac{1}{\sqrt{8\pi}}$.

25. Consider the following Linear Programming Problem

$$\begin{array}{ccll} \text{max} & 3x_1 + 2x_2 & & \\ \text{such that} & 2x_1 + x_2 + x_3 & = & 100 \\ & x_1 + x_2 + x_4 & = & 80 \\ & x_1 + x_5 & = & 40 \\ & x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

The basic variables at the point (20,60) are

 (a) X_1, X_3, X_5.
 (b) X_1, X_2, X_5.
 (c) X_2, X_3, X_4.
 (d) X_1, X_4, X_5.

PART B

- There are **17** questions in this part. Answer as many as you can.
- The maximum you can score is **50**. Marks are indicated against each question.
- The answers should be written in the separate answer script provided to you.

1. The probability density function of a random variable is

$$f(x) = \begin{cases} ax^2 \exp\{-kx\}, \ 0 \le x < \infty, \ a, k \ > 0 \\ 0, \ \text{otherwise.} \end{cases}$$

Given the constant $k > 0$, (i) find a. (ii) find the modal value of X.

[6 marks]

2. Suppose X and Y are the times of receipt of two signals with uniform distribution on [0,T]. Further suppose that the channel gets jammed if the time difference in the receipt of the two signals is less than τ, which is known. What is the probability that the channel will be jammed?

[6 marks]

3. $X_1, X_2, \ldots$ are independent and identically distributed random variables given by

$$X_i = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p \end{cases}$$

where p itself a random variable taking two values a and b, and $0 < a < b < 1$. Let $P\left(p = a\right) = \theta, \ 0 < \theta < 1$ and let $S_n = \sum_{i=1}^{n} X_i$.

(a) Compute $\phi_n(r) = P\left(p = a | S_n = r\right)$ for $r = 0, 1, 2, 3, \cdots, n$.

(b) Show that $\phi_n(r) > \frac{1}{\left(\frac{1-a}{1-b}\right)^n \left(\frac{1-\theta}{\theta}\right)+1}$.

(c) Find $E\left(p | S_n = r\right)$. **[6 marks]**

4. Let the random variable Y have exponential distribution with pdf:

$$f(y;\theta) = \begin{cases} (1/\theta)\exp(-y/\theta), & y \geq 0,\ \theta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $X = [Y]$, the integer part of Y.

(a) Determine how X is distributed.

(b) Show that X and $Y - X$, the fractional part of Y, are statistically independent.

[6 marks]

5. Consider the sampling design with $N = 7$ and $n = 3$

n			$p(s)$
U_1	U_2	U_4	1/7
U_2	U_3	U_5	1/7
U_3	U_4	U_6	1/7
U_4	U_5	U_7	1/7
U_5	U_6	U_1	1/7
U_6	U_7	U_2	1/7
U_7	U_1	U_3	1/7

Compute the inclusion probabilities π_i and π_{ij}, $i, j = 1, \cdots, 7$, $i \neq j$. How does this design compare with an $SRSWOR$ design ($N = 7$, $n = 3$)?

[8 marks]

6. If $X \sim N\left(\mathbf{0}, \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\right)$

(a) Find the distribution of $Z = \begin{pmatrix} X_1 - X_2 \\ X_2 - X_3 \\ X_3 - X_1 \end{pmatrix}$

(b) Let $Y_1 = X_1 + X_2 + X_3, Y_2 = X_1 + X_2 - X_3$. Find the conditional expectation and variance of Y_1 given $Y_2 = 2$. **[6 marks]**

7. Let $X_1, X_2, \ldots, X_n$ be i.i.d Binomial (k, θ). Find the uniformly minimum variance unbiased estimator for the probability of exactly one success.

[6 marks]

8. Let X be a Binomial $(N, 1/2)$ random variable where N, the number of trials is unknown, $N \in \{1, 2, \ldots\}$. To estimate N based on a single observation the following two confidence sets were considered:
(i) $\{X, X+1, X+2, \ldots\}$
(ii) $\{X, X+1, X+2, \ldots, 2X\}$.
Which of the two confidence sets will you prefer? Justify your choice.

[8 marks]

9. X is a discrete random variable on $A = \{0, 1, \ldots, 5\}$ with probability mass function $P_\theta(X = x)$, $x \in A$, $\theta \in \{\theta_0, \theta_1\}$ given below

x	0	1	2	3	4	5
$P_{\theta_0}(X = x)$	1/6	1/6	1/6	1/6	1/6	1/6
$P_{\theta_1}(X = x)$	1/12	1/12	1/4	1/4	4/15	1/15

Based on a single observation on X, the hypothesis to be tested are

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta = \theta_1.$$

(a) Derive a likelihood ratio test of size 0.2.

(b) Find the value of the power function of the proposed test under the alternative.

(c) If X=2 is observed, find the p-value of the proposed procedure and state your conclusion.

(d) What is the critical region of the likelihood ratio test at $\alpha = 0.5$ level of significance? **[10 marks]**

10. E_i, $i = 1, 2, 3$ are three independent events such that the probability that only E_i occurs is p_i. Show that the probability q that none of E_1, E_2, E_3 occur is a root of the equation $(q + p_1)(q + p_2)(q + p_3) = q^2$.

[6 marks]

11. A square matrix B is said to be idempotent if $B^2 = B$. Let I be the identity matrix of the same order as B.

(a) Show that $I + B$ is nonsingular.

(b) Show that $I - B$ is nonsingular if and only if $B = 0$.

[6 marks]

12. The value of Y is estimated for $X = x_0$ from the linear regression of Y on X. Let this estimated value of Y be y_0. The value of X for $Y = y_0$ is estimated from the linear regression of X on Y. Let this estimated value of X when $Y = y_0$ be x_0^*. Compare x_0 and x_0^*. Interpret the answer. **[6 marks]**

13. To compare the effects of three treatments A, B and C, the experimental field was split into three homogeneous blocks B_1, B_2 and B_3. Treatment A was given to all three blocks, treatment B was given to block B_1 and treatment C was given to blocks B_2 and B_3.

(a) Verify whether the resulting block design is (i) complete (ii) balanced (iii) connected (iv) orthogonal.

(b) Can the two treatments B and C be compared? Justify.

[10 marks]

14. Let $\{X_n\}$ be a Markov chain on $\{1, 2, \ldots, M\}$. The conditional distribution of X_{n+1} given $X_n = j,\ j = 1, 2, \ldots, (M-1)$ is discrete uniform on $\{j+1, \ldots, M\}$ and when $X_n = M$, X_{n+1} is equal to 1 with probability one. Obtain the mean time to return for each state $j = 1, 2, \ldots, M$.

[**8 marks**]

15. Let $X = \{\underline{x} :\ A\underline{x} = b,\ \underline{x} \geq 0\}$, where A is $m \times n$ matrix of rank m. Let $\underline{x}$ be a feasible solution, $\underline{x} = (x_1, \ldots, x_q, x_{q+1}, \ldots, x_n)'$ whose first q components, $x_1, \ldots, x_q$ are positive and next $n-q$ components $x_{q+1}, \ldots, x_n$ are zero. Assume that $\mathbf{a}_1, \ldots, \mathbf{a}_q$, the columns of A corresponding to $x_1, \ldots, x_q$, are dependent. Explain how you would construct feasible points $\underline{x}'$ and $\underline{x}''$ such that $\underline{x}$ is a convex combination of $\underline{x}'$ and $\underline{x}''$.

[**6 marks**]

16. A company manufactures three products A, B and C. The unit profit from making A is 3, B is 1 and C is 5. The amount of labour (in hours) required to make one unit of product A is 6 hours, one unit of product B is 3 hours, one unit of product C is 5 hours. The amount of material required to make one unit of product of A is 3 units, that of product B is 4 units and that of product C is 5 units. Total amount of labour hours available is 45 hours and total amount of material available is 30 units. The company wants to maximize its profit.

    (a) Formulate the problem as a linear programming problem.

    (b) Suppose the unit profit from B is increased from 1 to 4. What happens to the optimal solution?

    (c) A new product D with unit profit 5, labour requirement 3 hours and material requirement 4 units is planned to be introduced. Is it profitable for the company to produce D? [**10 marks**]

17. Consider the following initial problem

$$P = \min\ cx$$

$$\text{such that } Ax = b,\ x \geq 0.$$

Suppose P has a finite optimal solution. Show by Duality that the problem

$$P' = \min\ cx$$

$$\text{such that } Ax = b',\ x \geq 0.$$

can not be unbounded, no matter what value b' might take. [**8 marks**]