PART A

Each question carries 1 mark. $\frac{1}{3}$ mark will be deducted for each wrong answer. There will be no penalty if the question is left unanswered.

The set of real numbers is denoted by $\mathbb{R}$, the set of complex numbers by $\mathbb{C}$, the set of rational numbers by $\mathbb{Q}$ and the set of integers by $\mathbb{Z}$.

1. Let X be an inner-product space over the complex field $\mathbb{C}$. Which of the following statements are not equivalent to any other three statements? For x, $y \in X$

   (a) $x \perp y$, i.e., x is orthogonal to y.

   (b) $||x+y||^2 = ||x||^2 + ||y||^2$, where $||\cdot||$ is the norm on X induced by the inner product of X.

   (c) $||x+\alpha y|| = ||x-\alpha y||$ for all $\alpha \in \mathbb{C}$.

   (d) $||x+\alpha y|| \geq ||x||$ for all $\alpha \in \mathbb{C}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$. Then

   (I) if f is a differentiable function, then f is absolutely continuous.

   (II) if f is an absolutely continous function then f is uniformly continuous.

   (III) if f is a differentiable function then f is uniformly continuous.

   (a) Only (I) is true. (b) (II) and (III) are true.
   (c) (I) and (III) are true. (d) Only (II) is true.

3. The number of roots of $z^9 + z^5 + 8z^3 + 2z + 1 = 0$ between the circles $|z| = 1$ and $|z| = 2$ are
   (a) 3. (b) 4. (c) 5. (d) 6.

4. $T : \mathbb{R}^5 \to \mathbb{R}^5$ is a linear transformation with a minimal polynomial $(X^2+1)^2$. Then

   (a) there exists a vector v such that $T(v) = v$.

   (b) there exists a vector v such that $T(v) = -v$.

   (c) T must be singular.

   (d) such a linear transformation is not possible.

5. Let $f(x) = \begin{cases} \frac{1-x^2}{2} & \text{if } x < 0 \\ \frac{1+x^2}{2} & \text{if } x > 0. \end{cases}$ Then the derivative of f
(a) does not exist. (b) is $|x|$. (c) is x. (d) is $-|x|$.

6. How many automorphisms are there on $\mathbb{Z}_9 \times \mathbb{Z}_{16}$?
(a) 144. (b) 72. (c) 48. (d) 16.

7. For the function $f(z) = \sin\left(\frac{1}{z}\right)$, $z = 0$ is

(a) an essential singularity.
(b) a branch point.
(c) a removable singularity.
(d) a simple pole.

8. Let V be the set of all homogeneous polynomials of degree d in n-variables over a field F. Then $\dim V_F$ is
(a) $\binom{n}{d}$. (b) $\binom{d}{n}$. (c) $\binom{n+d-1}{d-1}$. (d) $\binom{n+d}{d}$.

9. The space l_p is a Hilbert space if and only if
(a) $p < 1$. (b) p is even. (c) $p = \infty$. (d) $p = 2$.

10. Let $\frac{dx}{dt} = f(x, y)$, $\frac{dy}{dt} = g(x, y)$, where f and g are C^1 functions of their arguments. The solution curves of this system of equations in the xy-plane
(I) are directed curves.
(II) do not intersect at any point.
(III) do intersect only at critical points.
Which of the above statements are true?
(a) (I) and (II). (b) (I) and (III).
(c) (II) and (III). (d) only (I).

11. The Banach space $(l^1, ||\cdot||_1)$ is

(a) reflexive and locally compact.
(b) reflexive but not locally compact.
(c) not reflexive but locally compact.
(d) neither reflexive nor locally compact.

12. The type of critical point $(0,0)$ of the system
$\dfrac{dx}{dt} = x + y + x^2 y, \quad \dfrac{dy}{dt} = 3x - y + 2xy^3$ is
(a) a saddle point. (b) a stable node.
(c) asymptotically stable. (d) not a simple critical point.

13. $\sum_{j=1}^{7} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, 6$ is a set of 6 real equations in seven variables. The condition that the system have infinitely many solutions is

(a) rank of $[a_{ij}]$ = rank of $[a_{ij}, b_i]$ where $[a_{ij}, b_i]$ is the augmented matrix.

(b) a 6×6 minor of $[a_{ij}]$ must have a non-zero determinant.

(c) a 6×6 minor of $[a_{ij}]$ must have a zero determinant.

(d) the matrix $[a_{ij}]$ must have rank 6.

14. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and a be a non-zero real number. Suppose $aE = \{ax : x \in E\}$. Then the Lebesgue measure $m(aE)$ is equal to
(a) $m(E)$. (b) $a\, m(E)$. (c) $|a|\, m(E)$. (d) none of these.

15. The locus represented by the equation in complex variables $z = x + iy$, $|z-2| + |z+4| = 10$ is
(a) a circle. (b) a parabola. (c) a straight line. (d) an ellipse.

16. Suppose G is a finite group and N is a normal subgroup and H is a subgroup. Let $n = |G/H|$. If $gcd(n, O(N)) = 1$, then
(a) $H \subseteq N$. (b) $N \subseteq H$. (c) $H = N$. (d) $O(H)$ divides $O(N)$.

17. Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be given by
$$f((a,b,c,d)) = (3a - 2b + c + d, 3a - 7b - 7c + 8d, a + b + 3c - 2d).$$
Then

(a) f is onto but not $1-1$.

(b) f is $1-1$ but not onto.

(c) f is both $1-1$ and onto.

(d) f is neither $1-1$ nor onto.

18. The partial differential equation $u_{xx} - xu_{yy}$ is

(a) elliptic, $x > 0$. (b) hyperbolic, $x < 0$.
(c) hyperbolic, $x > 0$. (d) none of these.

19. Let X be a compact Hausdorff space. Then

(a) X is metrizable and separable.
(b) X is metrizable but need not be separable.
(c) X is normal but need not be metrizable.
(d) X is completely regular but need not be normal.

20. Let X be a connected metric space and let $Y \subset X$.

(a) If Y is open, then Y is connected.
(b) If Y is closed, then Y connected.
(c) If Y is compact, then Y is connected.
(d) All of (a), (b), (c) are false.

21. Let $f : [0,2] \to [0,1]$ be any continuously differentiable function. Then

(a) $|f'(x)| \leq 1 \quad \forall \quad x \in [0,2]$.
(b) $|f'(x)| \leq 2 \quad \forall \quad x \in [0,2]$.
(c) $|f'|$ is bounded but need not be bounded by 2.
(d) $|f'|$ need not be bounded.

22. Let d be the Euclidian metric and $\hat{d}$ be another metric on $\mathbb{R}$. Let $A \subset \mathbb{R}$ be closed and bounded with respect to $\hat{d}$. Then

(a) A is compact.
(b) A is compact only if $\hat{d} = d$.
(c) A is compact if the topologies induced by d and $\hat{d}$ are the same.
(d) A is compact if there exists $\alpha > 0, \beta > 0$ such that $\alpha\, d(x,y) \leq \hat{d}(x,y) \leq \beta\, d(x,y) \ \forall \ x, y \in \mathbb{R}$.

23. Let $X \equiv \mathbb{R}$ and $\mathcal{T} = \{U \mid X - U \text{ is countable}\} \cup \{\phi\}$. Then $(X, \mathcal{T})$ is

(a) not connected but compact.
(b) connected and compact.
(c) connected but not compact.
(d) neither connected nor compact.

24. Let $f(x) = x^2$, $g(x) = x|x|$ be defined in $[-1, 1]$, and let W be the Wronskian of f and g on $[-1, 1]$. Then on $[-1, 1]$

(a) f and g are linearly independent and $W = 0$.

(b) f and g are linearly dependent and $W = 0$.

(c) f and g are linearly independent and $W \neq 0$.

(d) f and g are linearly dependent and $W \neq 0$.

25. $F(z-xy, x^2+y^2) = 0$ is the solution of the partial differential equation

(a) $yz_x - xz_y = y^2 - x^2$.
(b) $yz_x + xz_y = y^2 - x^2$.
(c) $yz_x + xz_y = y^2 + x^2$.
(d) $yz_x - xz_y = y^2 + x^2$

PART B

Answer any **ten** questions. Each question carries 5 marks.

1. Evaluate the integral $\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x, \quad a^2>0, b^2>0$ using Laplace's inversion formula.

2. The cardinality of G, $|G| = pq$, where p, q are prime $p < q$. Show that if $q \not\equiv\ 1\text{mod}(p)$ then G has at least two non-trivial normal abelian subgroups.

3. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ (need not be continuous) whose graph $\{(x, f(x)) : x \in \mathbb{R}\}$ is dense in $\mathbb{R}^2$? Justify your answer.

4. Show that the sequence of functions $f_n(x) = n^2 x^n (1 - x)$ defined on $I = [0, 1]$ converges point wise to zero on I but not uniformly.

5. Let G be a group with a topology on it such that for every $x \in G$ the map from $G \to G$, $y \mapsto xy$ is a homeomorphism. Let H be a subgroup. Suppose H is open. Show that H is also closed.

6. Consider the Banach space $C[-1,1]$ of all real valued continuous functions defined on $[-1,1]$ with the norm $\|f\|_\infty = \sup\limits_{-1\leq x\leq 1} |f(x)|$ where $f \in C[-1,1]$. For $f \in C[-1,1]$, denote by $\tilde{f}(x) = f(|x|)$, $x \in [-1,1]$. Define $T : C[-1,1] \to C[-1,1]$ by $T(f) = \tilde{f}$. Show that T is a bounded linear operator on $[-1,1]$. Determine the value of $\|T\|$.

7. Determine all real numbers $L > 1$ so that the boundary value problem

$$x^2y''(x) + y(x) = 0, \quad 1 < x < L; \quad y(1) = y(L) = 0$$

has a non-trivial solution.

8. Using Green's function, find the solution of the ordinary differential equation $y'' + y = 1, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0.$

9. Show that $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$, by contour integration along a closed contour consisting of the semi-circular arc of radius R passing through $(R, 0)$, $(-R, 0)$ and the real axis from $(-R, 0)$ to $(R, 0)$ including the real axis.

10. Show that a finite field cannot be algebraically closed.

11. Discuss whether the transformation $Q = \arctan\left(\frac{\alpha q}{p}\right)$, $p = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$ is canonical, where α is an arbitrary constant.

12. Let A be a 5×5 matrix with a minimal polynomial $(x-1)(x+2)^2$. What are the possible Jordan canonical forms for A?

13. Determine the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ by computing its echelon form.

14. Let $p : \mathbb{R} \to \mathbb{R}$ be a non-constant real polynomial. If $a_n = p(n)$ for $n = 0, 1, 2, \ldots,$ then determine the radius of convergence of the complex power series $\sum_{n=0}^{\infty} a_n z^n$.

15. Find the general integral of the first order partial differential equation $xu_x + (1+y)u_y = x(1+y) + xu$.

16. Solve the following Cauchy problem $z = xp + yq - p^2q$ with the initial data $x_0(s) = s$, $y_0(s) = 2$, $z_0(s) = s + 1$.

17. Investigate for solvability the integral equation

$$\phi(x) - \lambda \int_0^{2\pi} |x - \pi| \, \phi(t) dt = x.$$

18. Consider the following Linear Programming problem

$$\begin{array}{rl} P : \max & 2x_1 + x_2 - x_3 \\ \text{such that} & x_1 + 2x_2 + x_3 \leq 8 \\ & -x_1 + x_2 - 2x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

The optimal solution to this problem has $x_1 = 8$ and $x_5 = 12$ in the basis. Now if you have to choose between increasing the right hand sides of the first constraint and second constraint, which one would you choose? Explain your choice. What is the effect of this increase on the optimal value of the objective function?

19. Consider the following LP

$$\begin{aligned} P:\min \quad & -x_2 \\ \text{such that} \quad & x_1 + x_2 \geq 2 \\ & -x_2 \geq -4 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solve P graphically to obtain the primal optimal solution. State the dual D and solve it graphically. Write down the complementary slackness condition for the the problem and illustrate the same for the primal and dual variables here.

20. Consider the following transportation problem with 3 origins and 4 destinations and the c_{ij} cost matrix given by

Origin ↓ Destination →	1	2	3	4	s_i
1	1	2	3	4	7
2	2	2	4	2	7
3	3	4	3	4	6
d_j	6	8	6	5	

(a) Is the problem balanced? If not, reformulate it as a balanced transportation problem.

(b) Find an initial basic feasible solution using any method.