University of Hyderabad,
Entrance Examination, 2008
Ph.D. (Mathematics/Applied Mathematics)

Hall Ticket No.							

Time: 2 hours Max. Marks: 75
Part A: 25 Marks
Part B: 50 Marks

Instructions

1. Calculators are not allowed.
2. Part A carries 25 marks. Each correct answer carries **1 mark** and each wrong answer carries **minus one third mark**. So do not gamble. If you want to change any answer, cross out the old one and circle the new one. Over written answers will be ignored.
3. Part B carries 50 marks. Instructions for answering Part B are given at the beginning of Part B.
4. Do not detach any pages from this answer book. It contains **15** pages in addition to this top page. Pages **14** and **15** are for rough work.

Answer Part A by **circling** the correct letter in the array below:

1	a	b	c	d
2	a	b	c	d
3	a	b	c	d
4	a	b	c	d
5	a	b	c	d
6	a	b	c	d
7	a	b	c	d
8	a	b	c	d
9	a	b	c	d
10	a	b	c	d
11	a	b	c	d
12	a	b	c	d
13	a	b	c	d
14	a	b	c	d
15	a	b	c	d
16	a	b	c	d
17	a	b	c	d
18	a	b	c	d
19	a	b	c	d
20	a	b	c	d
21	a	b	c	d
22	a	b	c	d
23	a	b	c	d
24	a	b	c	d
25	a	b	c	d

Part A

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) = \min(1, x, x^3)$. Then

   (a) f is continuous but not differentiable on $\mathbb{R}$.

   (b) f is continuous and differentiable on $\mathbb{R}$.

   (c) f is not continuous but differentiable on $\mathbb{R}$.

   (d) f is neither continuous nor differentiable on $\mathbb{R}$.

2. Let G be an infinite cyclic group. If f is an automorphism of G, then

   (a) $f^n \neq Id_G$ for any $n \in \mathbb{N}$.

   (b) $f^2 = Id_G$.

   (c) $f = Id_G$.

   (d) there exists an $n \in \mathbb{N}$ such that $f(x) = x^n$, for all $x \in G$.

3. Let G be a group of order 10 . Then

   (a) G is an abelian group.

   (b) G is a cyclic group.

   (c) there is a normal proper subgroup.

   (d) there is a subgroup of order 5 which is not normal.

4. For each $\alpha \in I$, let X_α be a non-empty topological space such that the product space $\prod_{\alpha \in I} X_\alpha$ is locally compact. Then

   (a) X_α must be compact except for finitely many α.

   (b) X_α must be a singleton except for finitely many α.

   (c) each X_α must be compact.

   (d) the indexing set I must be countable.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the set $\{x \in \mathbb{R} : f \text{ is continuous at } x\}$ is always

   (a) a G_δ set. (b) an F_σ set.

   (c) an open set. (d) a closed set.

6. Let $A = \mathbb{R} \times \mathbb{R}$ and $B = Q \times Q$. Two distinct points in $A \setminus B$ can be joined together within $A \setminus B$

   (a) always by a line segment.

   (b) always by a smooth path.

   (c) not always by a smooth path but always by a continuous path.

   (d) cannot be joined together always by a continuous path.

7. Let G be a group of order 255. Then

   (a) the number of Sylow - 3 subgroups cannot be more than 1.

   (b) the number of Sylow - 11 subgroups is at least 1.

   (c) the number of Sylow - 3 subgroups is 1 or 85.

   (d) the number of Sylow - 5 subgroups is 51.

8. The number of ideals in the ring $\dfrac{\mathbb{R}[x]}{(x^2-1)}$ is

   (a) 1. (b) 2. (c) 3. (d) 4.

9. All the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ lie in the disc

   (a) $|\lambda + 1| \leq 1$. (b) $|\lambda - 1| \leq 1$.
   (c) $|\lambda + 1| \leq 2$. (d) $|\lambda - 1| \leq 2$.

10. For the ordinary differential equation $\sin(x)y''(x) + y'(x) + y(x) = 0$,

    (a) every point is an ordinary point.

    (b) every point is a singular point.

    (c) $x = n\pi$ is a regular singular point.

    (d) $x = n\pi$ is an irregular singular point.

11. If in a group, an element a has order 65, then the order of a^{25} is
    (a) 5. (b) 12. (c) 13. (d) 65.

12. The number of subfields of $\mathbb{F}_{2^{27}}$ (distinct from $\mathbb{F}_{2^{27}}$ itself) is
    (a) 1. (b) 2. (c) 3. (d) 4.

13. The number of Jordan canonical forms for a 5×5 matrix with minimal polynomial $(x-2)^2(x-3)$ is
    (a) 1. (b) 2. (c) 3. (d) 4.

14. The number of degrees of freedom of a rigid cube moving in space is
    (a) 1. (b) 3. (c) 5. (d) 6.

15. Let $A \subset \mathbb{R}$ be a measurable set. Then

    (a) If A is dense then the Lebesgue measure of A is positive.

    (b) If the Lebesgue measure of A is zero then A is nowhere dense.

    (c) If the Lebesgue measure of A is positive then A contains a nontrivial interval.

    (d) All of (a), (b), (c) are false.

16. The equation $u_{xx} + x^2 u_{yy} = 0$ is

    (a) elliptic.

    (b) elliptic everywhere except on $x = 0$ axis.

    (c) hyperbolic.

    (d) hyperbolic everywhere except on $x = 0$ axis.

17. The solution of the Laplace equation in spherical polar co-ordinates (r, θ, ϕ) is
    (a) $\log(r)$. (b) r. (c) $1/r$. (d) r and $1/r$.

18. A particle moves in a circular orbit in a force field $F(r) = -K/r^2$, $(K > 0)$. If K decreases to half its original value then the particle's orbit
    (a) is unchanged. (b) becomes parabolic.
    (c) becomes elliptic. (d) becomes hyperbolic.

19. Let $T : X \to Y$ be a linear map between normed spaces over $\mathbb{C}$. Then the minimum requirement ensuring the continuity of T is
    (a) X is finite dimensional. (b) X and Y are finite dimensional.
    (c) $Y = \mathbb{C}$. (d) Y is finite dimensional.

20. Let H be a Hilbert space. Which of the following is true?

    (a) H is always separable.

    (b) If H has an orthogonal Schauder basis, then H is separable.

    (c) If H is separable, then H is locally compact.

    (d) If H has a countable Hamel basis, then H is finite dimensional.

21. For each $n \in \mathbb{N}$, let $f_n : [0,1] \to [0,1]$ be a continuous function and let $f : [0,1] \to [0,1]$ be defined as $f(x) = \limsup_{n\to\infty} f_n(x)$. Then

    (a) f is continuous and measurable.

    (b) f is continuous but need not be measurable.

    (c) f is measurable but need not be continuous.

    (d) f need not be either continuous or measurable.

22. Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic and let $A = \{x \in \mathbb{R} : f(x) = g(x)\}$. The <u>minimum</u> requirement for the equality $f = g$ is

    (a) A is uncountable. (b) A has a positive Lebesgue measure.

    (c) A contains a nontrivial interval. (d) $A = \mathbb{R}$.

23. The critical point of the system $x'(t) = -y + x^2, \quad y'(t) = x$ is

    (a) a stable center. (b) unstable.

    (c) an asymptotically stable node. (d) an asymptotically stable spiral.

24. An example of a subset of $\mathbb{N}$ which intersects every set of form $\{a + nd : n \in \mathbb{N}\}$, $a, d \in \mathbb{N}$, is

    (a) $\{2k : k \in \mathbb{N}\}$. (b) $\{k^2 : k \in \mathbb{N}\}$.

    (c) $\{k + k! : k \in \mathbb{N}\}$. (d) $\{k + k^2 : k \in \mathbb{N}\}$.

25. The characteristic number of the integral equation $\phi(x) - \lambda \int_0^{2\pi} \sin(x)\sin(t)\phi(t)\ dt = 0$ is

    (a) π. (b) $\frac{1}{\pi}$. (c) 2π. $\frac{1}{2\pi}$.

Part B

Answer any **Ten** questions

1. Let f be a map from $\mathbb{R}$ to $\mathbb{R}$ such that $f(a+b) = f(a)f(b)$. If $f \neq 0$ and it is continuous at 0 then show that there exists a nonzero $c \in \mathbb{R}$ such that $f(x) = c^x$ for all $x \in \mathbb{R}$.

2. Give an entire function whose image omits only the value 2π. Also find a Möbius map whose only fixed point is 2π.

3. Let $f(z) = z^6 - 5z^5 + 2z^4 + 1$ and $K = \{z \in \mathbb{C} : |z - 2i| \leq 1\}$. Show that $\min\{|f(z)| : z \in K\}$ is attained at some point on the boundary of K.

4. Let $f : W \to \mathbb{R}^3$ be a linear transformation given by $f(\lambda_1 v_1 + \lambda_2 v_2) = (\lambda_1, \lambda_2, 0)$ where W is the space generated by the vectors $v_1 = (1, 1, -1)$ and $v_2 = (1, -1, 1)$. Describe how you would extend f to $\mathbb{R}^3$ so that the determinant of f is 1. Define such an extended f.

5. Consider the Banach space ℓ_1 of all complex sequences $\{\alpha_n\}$ such that $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ with the norm $|| \{\alpha_n\} ||_1 = \sum_{n=1}^{\infty} |\alpha_n|$. Let $\{\lambda_n\}$ be a sequence of complex numbers such that $\{\lambda_n \alpha_n\} \in \ell_1$ for all $\{\alpha_n\} \in \ell_1$. Define $T : \ell_1 \to \ell_1$ by $T(\{\alpha_n\}) = \{\lambda_n \alpha_n\}$. If T is a bounded linear operator on ℓ_1 then show that $\{\lambda_n\}$ is bounded. In this case what will be the value of $||T||$?

6. Determine the smallest m such that the field with 5^m elements has a primitive 12^{th} root of 1.

7. Let $A = \{\alpha \in \mathbb{R} \mid a\alpha^2 + b\alpha + c = 0 \text{ for some integers } a, b, c\}$. Then prove that A is a countably infinite set.

8. Let $\mathbb{R}^{\mathbb{N}}$ be the set of all sequences of real numbers. Two members (a_n) and (b_n) are said to be asymptotic if $\limsup_{n\to\infty}(|a_n - b_n|) = 0$; they are said to be proximal if $\liminf_{n\to\infty}(|a_n - b_n|) = 0$. Prove that asymptoticity is an equivalence relation on $\mathbb{R}^{\mathbb{N}}$ where as proximality is not. Give an example of a proximal pair that is not asymptotic.

9. Define a topology $\mathcal{T}$ on $\mathbb{R}$ by declaring a subset $U \subset \mathbb{R}$ to be open if $U = \phi$ or $0 \in U$. Describe all finite subsets of $\mathbb{R}$ which are dense in $(\mathbb{R}, \mathcal{T})$. Give a basis of $(\mathbb{R}, \mathcal{T})$ each of whose element is a finite set.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with a bounded derivative. Define $f_n(x) = f\left(x + \frac{1}{n}\right)$. Show that f_n converges uniformly on $\mathbb{R}$ to f.

11. Let $f_n(x) = x^n$ for $0 \le x \le 1$. Find the pointwise limit f of the sequence $\{f_n\}$. Prove that $\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x$. Is the convergence uniform?

12. Find the extremal of the functional $J[y] = \int_0^1 \left(x + 2y + \frac{y'^2}{2}\right)\,\mathrm{d}x$, $y(0) = 0$, $y(1) = 0$. Also test for extrema.

13. Construct the Green's function for the boundary value problem $y'' + y = 0$ subject to the boundary conditions $y(0) + y'(\pi) = 0$, $y'(0) - y(\pi) = 0$.

14. Find the complete integral of $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$.

15. Solve the integral equation $\phi(x) - \lambda \int_0^{2\pi} |x - t| \sin(x)\phi(t) \, \mathrm{d}t = x$.

Rough Work

Rough Work