University of Hyderabad,
Entrance Examination, 2010
Ph.D. (Mathematics/Applied Mathematics/OR)

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Hall Ticket No.	

Time: 2 hours

Max. Marks: 75
Part A: 25
Part B: 50

Instructions

1. Calculators are not allowed.
2. Part A carries 25 marks. Each correct answer carries **1 mark** and each wrong answer carries −0.33mark. So do not gamble. If you want to change any answer, cross out the old one and circle the new one. Over written answers will be ignored.
3. Part B carries 50 marks. Instructions for answering Part B are given at the beginning of Part B.
4. Use a separate booklet for Part B.

Answer Part A by **circling** the correct letter in the array below:

1	a	b	c	d
2	a	b	c	d
3	a	b	c	d
4	a	b	c	d
5	a	b	c	d
6	a	b	c	d
7	a	b	c	d
8	a	b	c	d
9	a	b	c	d
10	a	b	c	d
11	a	b	c	d
12	a	b	c	d
13	a	b	c	d
14	a	b	c	d
15	a	b	c	d
16	a	b	c	d
17	a	b	c	d
18	a	b	c	d
19	a	b	c	d
20	a	b	c	d
21	a	b	c	d
22	a	b	c	d
23	a	b	c	d
24	a	b	c	d
25	a	b	c	d

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PART A

Each question carries 1 mark. 0.33 mark will be deducted for each wrong answer. There will be no penalty if the question is left unanswered.
The set of real numbers is denoted by $\mathbb{R}$, the set of complex numbers by $\mathbb{C}$, the set of rational numbers by $\mathbb{Q}$ and the set of integers by $\mathbb{Z}$.

1. Let V be a real vector space and $S = \{v_1, v_2, \ldots, v_k\}$ be a linearly independent subset of V. Then

   (a) dim $V = k$.

   (b) dim $V < k$.

   (c) dim $V \geq k$.

   (d) nothing can be said about dim V.

2. The value of $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$ is

   (a) 0. (b) $\frac{1}{2}$. (c) 1. (d) $\frac{3}{2}$.

3. Consider the function $f(x)$ on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3, & \text{if } x^2 \leq 1 \\ x, & \text{if } x^2 \geq 1. \end{cases}$$

   Then

   (a) f is continuous at each point of $\mathbb{R}$.

   (b) f is continuous at each point of except at $x = \pm 1$.

   (c) f is differentiable at each point of $\mathbb{R}$.

   (d) f is not continuous at any point of $\mathbb{R}$.

4. Let $f(x, y)$ be defined on $\mathbb{R}^2$ by $f(x, y) = |x| + |y|$. Then

   (a) the partial derivatives of f at $(0, 0)$ exist.

   (b) f is differentiable at $(0, 0)$.

   (c) f is continuous at $(0, 0)$.

   (d) none of the above hold.

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5. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous taking values in $\mathbb{Q}$, the set of rational numbers. Then

(a) f is strictly monotone.

(b) f is unbounded.

(c) f is differentiable.

(d) the image of f is infinite

6. For the set $\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$, the element $\frac{1}{4}$ is

(a) both an element in the set and a limit point of the set.

(b) neither an element in the set nor a limit point.

(c) an element in the set, but not a limit point.

(d) a limit point of the set, but not an element in the set.

7. If $|\tan z| = 1$, then

(a) $\text{Re } z = \frac{\pi}{4} + \frac{n\pi}{2}$.

(b) $\text{Re } z = \frac{\pi}{4} + n\pi$.

(c) $\text{Re } z = \frac{\pi}{2} + n\pi$.

(d) $\text{Re } z = \frac{\pi}{2} + \frac{n\pi}{2}$.

8. The number of zeroes of $z^9 + z^5 - 8z^3 + 2z + 1$ in the annular region $1 \le |z| \le 2$ are

(a) 3. (b) 6. (c) 9. (d) 14.

9. The residue of $f(z) = \cot z$ at any of its poles is

(a) 0. (b) 1. (c) $\sqrt{2}$. (d) $2\sqrt{3}$.

10. Let (X, d) be a metric space and $A \subset X$. Then A is totally bounded if and only if

    (a) every sequence in A has a Cauchy subsequence.

    (b) every sequence in A has a convergent subsequence.

    (c) every sequence in A has a bounded subsequence.

    (d) every bounded sequence in A has a convergent subsequence.

11. Suppose N is a normal subgroup of G. For an element x in a group, let $O(x)$ denote the order of x. Then

    (a) $O(a)$ divides $O(aN)$.

    (b) $O(aN)$ divides $O(a)$.

    (c) $O(a) \neq O(aN)$.

    (d) $O(a) = O(aN)$.

12. If G is a group such that it has a unique element a of order n. Then

    (a) $n = 2$.

    (b) n is a prime.

    (c) n is an odd prime.

    (d) $O(G) = n$.

13. Consider the ring $\mathbb{Z}$. Then

    (a) all its ideals are prime.

    (b) all its non-zero ideals are maximal.

    (c) $\mathbb{Z}/I$ is an integral domain for any ideal I of $\mathbb{Z}$.

    (d) any generator of a maximal ideal in $\mathbb{Z}$ is prime.

14. F_n denotes the finite field with n elements. Then

    (a) $F_4 \subset F_8$.

    (b) $F_4 \subset F_{12}$.

    (c) $F_4 \subset F_{16}$.

    (d) $F_4 \subset F_{32}$.

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15. Let A be an $n \times n$ matrix which is both Hermitian and unitary. Then

 (a) $A^2 = I$.
 (b) A is real.
 (c) The eigenvalues of A are $0, 1, -1$.
 (d) The minimal and characteristic polynomials are same.

16. For $0 < \theta < \pi$, the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

 (a) has real eigenvalues.
 (b) is symmetric.
 (c) is skew-symmetric.
 (d) is orthogonal.

17. Let $\{u, v\} \subset \mathbb{R}^3$ be a linearly independent set and let $A = \{w \in \mathbb{R}^3 : ||w|| = 1$ and $\{nu, v, w\}$ is linearly independent for some $n \in \mathbb{N}\}$. Then

 (a) A is a singleton.
 (b) A is finite but not a singleton.
 (c) A is countably finite.
 (d) A is uncountable.

18. A : A is a 5×5 complex matrix of finite order, that is, $A^k = I$ for some $k \in \mathbb{N}$, must be diagonalizable.
B : A diagonalizable 5×5 matrix must be of finite order.
Then

 (a) A and B are both true.
 (b) A is true but B is false.
 (c) B is true but A is false.
 (d) Both A and B are false.

19. Let $V = C[0,1]$ be the vector space of continuous functions on $[0,1]$. Let $||f||_1 := \int_0^1 |f(t)|dt$ and $||f||_\infty := \sup\{|f(t)| : 0 \leq t \leq 1\}$. Then

    (a) $(V, || \ ||_1)$ and $(V, || \ ||_\infty)$ are Banach spaces.

    (b) $(V, || \ ||_\infty)$ is complete but $(V, || \ ||_1)$ is not.

    (c) $(V, || \ ||_1)$ is complete but $(V, || \ ||_\infty)$ is not.

    (d) Neither of the spaces $(V, || \ ||_1)$, $(V, || \ ||_\infty)$ are complete.

20. The space l_p is a Hilbert space if and only if
    (a) $p > 1$. (b) p is even. (c) $p = \infty$. (d) $p = 2$.

21. Which of the following statements is correct?

    (a) On every vector space V over $\mathbb{R}$ or $\mathbb{C}$, there is a norm with respect to which V is a Banach space.

    (b) If $(X, ||.||)$ is a normed space and Y is a subspace of X, then every bounded linear functional f_0 on Y has a unique bounded linear extension f to X such that $||f|| = ||f_0||$.

    (c) Dual of a separable Banach space is separable.

    (d) A finite dimensional vector space is a Banach space with respect to any norm on it.

22. The critical point $(0,0)$ for the system $\frac{dx}{dt} = 2x, \frac{dy}{dt} = 3y$ is

    (a) a stable node

    (b) an unstable node.

    (c) a stable spiral.

    (d) an unstable spiral.

23. The set of linearly independent solutions of $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0$ is

    (a) $\{1, x, e^x, e^{-x}\}$.

    (b) $\{1, x, e^{-x}, xe^{-x}\}$.

    (c) $\{1, x, e^x, xe^x\}$.

    (d) $\{1, x, e^x, xe^{-x}\}$.

24. The set of all eigenvalues of the Sturm-Liouville problem

$$y'' + \lambda y = 0,\ y'(0) = 0,\ y'(\frac{\pi}{2}) = 0$$

is given by

(a) $\lambda \neq 0, \lambda = 2n, n = 1, 2, 3, \ldots.$

(b) $\lambda \neq 0, \lambda = 4n^2, n = 1, 2, 3, \ldots.$

(c) $\lambda = 2n, n = 0, 1, 2, 3, \ldots.$

(d) $\lambda = 4n^2, n = 0, 1, 2, 3, \ldots.$

25. A complete integral of $zpq = p^2q(x+q) + pq^2(y+p)$ is

(a) $z = ax + by - 2ab.$

(b) $xz = ax - by + 2ab.$

(c) $xz = by - ax + 2ab.$

(d) $xz = ax + by + 2ab.$

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PART B

Answer any **ten** questions. Each question carries 5 marks.

1. List all possible Jordan Canonical forms of a 7×7 real matrix whose minimal polynomial is $(x-1)^2(x-2)(x+3)$ and characteristic polynomial is $(x-1)^2(x-2)^2(x+3)^3$.

2. A is an $m \times n$ matrix and B is an $n \times m$ matrix, $n < m$, then prove that AB is never invertible.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function with the point at infinity as a pole. Show that f is a polynomial.

4. Show that $L^2([0,1]) \subseteq L^1([0,1])$ and the inclusion map $f \to f$ from $L^2[0,1]$ to $L^1[0,1]$ is a bounded linear operator.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex analytic function such that $f(f(z))$ for all $z \in \mathbb{C}$ with $|z| = 1$. Show that f is either constant or identity.

6. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial whose co-efficients satisfy $\sum_{i=0}^{n} \frac{a_i}{i+1} = 0$. Then $p(x)$ has a real root between 0 and 1.

7. Give an example of a decreasing sequence $\{f_n\}$ of measurable functions defined on a measurable set E of $\mathbb{R}$ such that $f_n \to f$ pointwise a.e. on E but $\int_E f \neq \lim_{n\to\infty} \int_E f_n$.

8. Show that all 3-Sylow subgroups in S_4 are conjugate.

9. Let α be algebraic over a field K such that the degree $[K(\alpha) : K]$ is odd. Show that $K(\alpha) = K(\alpha^2)$.

10. Let E be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, H be the hyperbola $xy = 1$ and P be the parabola $y = x^2$. Show that no two of these are homeomorphic.

11. Verify whether

$$Q_1 = q_1 q_2, \quad Q_2 = q_1 + q_2,$$

$$P_1 = \frac{p_1 - p_2}{q_1 - q_2} + 1, \quad P_2 = \frac{q_2 p_2 - q_1 p_1}{q_2 - q_1} - (q_2 + q_1),$$

is a canonical transformation for a system having two degrees of freedom.

12. Determine the Green's function for the boundary value problem

$$xy'' + y' = -f(x), \quad 1 < x < \infty$$
$$y(1) = 0,$$
$$\lim_{x\to\infty} |y(x)| < \infty$$

13. Find the critical point of the system

$$\frac{dx}{dt} = x + y - 2xy, \quad \frac{dy}{dt} = -2x + y + 3y^2,$$

and discuss its nature and stability.

14. Reduce the following partial differential equation to a canonical form and solve, if possible.

$$x^2 u_{xx} + 2xu_{xy} + u_{yy} = u_y.$$

15. Determine the two solutions of the equation $pq = 1$ passing through the straight line $C : x_0 = 2s, y_0 = 2s, z_0 = 5s$.

16. Let $\hat{x}$ denote the optimal solution to the following linear programming problem P1 :

$$\begin{array}{rrcl} & \min & c^T x & \\ s.t. & Ax & \geq & b \\ & x & \geq & 0, \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and $c, x \in \mathbb{R}^n$. Now, a new constraint $\alpha^T x \geq \beta$, where $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, is added to the feasible region and we get the following linear programming problem P2 :

$$\begin{array}{rrcl} & \min & c^T x & \\ s.t. & Ax & \geq & b \\ & \alpha^T x & \geq & \beta \\ & x & \geq & 0. \end{array}$$

Discuss about the optimality of $\hat{x}$ for the two cases (a) $\hat{x}$ is feasible to the new LP (P2) and (b) $\hat{x}$ is not feasible to the new LP (P2).

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17. Consider the following linear programming problem

$$\begin{aligned} \min \quad & x_1 + x_2 \\ s.t. \quad sx_1 + tx_2 &\geq 1 \\ x_1 &\geq 0 \\ x_2 & \quad \text{unrestricted.} \end{aligned}$$

Find conditions on s and t to make the linear programming problem have (a) multiple optimal solutions and (b) an unbounded solution.

18. Five employees are available for four jobs in a firm. The time (in minutes) taken by each employee to complete each job is given in the table below.

Person↓ Job→	1	2	3	4
1	24	18	32	18
2	20	-	29	24
3	28	22	30	30
4	18	24	-	16
5	23	-	27	29

The objective of the firm is to assign employees to jobs so as to minimize the total time taken to perform the four jobs. Dashes indicate a person cannot do a particular job.

(a) Formulate the above problem as a linear programming problem and state the Dual of the problem.

(b) What is the optimal assignment to the problem?