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University of Hyderabad,
Entrance Examination, 2010
Ph.D. (Statistics-OR)

Hall Ticket No.	

Time: 2 hours

Max. Marks: **75**
Part A: **25**
Part B: **50**

Instructions

1. Calculators are not allowed.
2. Part A carries 25 marks. Each correct answer carries **1 mark** and each wrong answer carries **− 0.33 mark**. If you want to change any answer, cross out the old one and circle the new one. Over written answers will be ignored.
3. Part B carries 50 marks. Instructions for answering Part B are given at the beginning of Part B.
4. Use a separate booklet for Part B.

Answer Part A by **circling** the correct letter in the array below:

1	a	b	c	d
2	a	b	c	d
3	a	b	c	d
4	a	b	c	d
5	a	b	c	d
6	a	b	c	d
7	a	b	c	d
8	a	b	c	d
9	a	b	c	d
10	a	b	c	d
11	a	b	c	d
12	a	b	c	d
13	a	b	c	d
14	a	b	c	d
15	a	b	c	d
16	a	b	c	d
17	a	b	c	d
18	a	b	c	d
19	a	b	c	d
20	a	b	c	d
21	a	b	c	d
22	a	b	c	d
23	a	b	c	d
24	a	b	c	d
25	a	b	c	d

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PART A

- Find the correct answer and mark it on the answer sheet on the **top page**.
- A right answer gets **1 mark** and a wrong answer gets **-0.33 mark**.

1. $X_1, ..., X_n$ are i.i.d random variables with absolutely continuous distribution function $F(x;\theta)$, then $-\sum_{i=1}^{n} logF(X_i;\theta)$ has

   (a) Normal distribution.

   (b) Beta distribution.

   (c) Gamma distribution.

   (d) Weibull distribution.

2. Let x be an observation from Bernoulli random variable taking values 0 and 1 with probabilities θ and $1-\theta$ respectively. If $\theta \in [\frac{1}{2}, \frac{3}{4}]$, the ML estimate of θ is

   (a) $\frac{x}{4}$.

   (b) $x + 2$.

   (c) $\frac{x+2}{4}$.

   (d) $\frac{x+3}{2}$.

3. Let $X_1, ..., X_n$ be a random sample from the Bernoulli distribution as described in Question 2, an unbiased estimator for θ^2 is

   (a) $\frac{\bar{X}-n\bar{X}^2}{2(n-1)}$.

   (b) $\frac{\bar{X}-n\bar{X}^2}{n-1}$.

   (c) $\frac{\bar{X}+n\bar{X}^2}{n-1}$.

   (d) $\frac{n\bar{X}^2-\bar{X}}{n-1}$.

4. Let T be Binomial random variable with parameters n and θ, $E\left[\binom{T}{r}\binom{n-T}{n-r}\right]$ for each $r = 0, 1, ..., n$ are equal to

   (a) $n(n-1)\theta^r(1-\theta)^{n-r}, \quad r = 0, 1, ..., n.$

   (b) $n\theta^r(1-\theta)^{n-r}, \quad r = 0, 1, ..., n.$

   (c) $\binom{n}{r}\theta^r(1-\theta)^{n-r}, \quad r = 0, 1, ..., n.$

   (d) $\frac{n}{r}\theta^r(1-\theta)^{n-r}, \quad r = 0, 1, ..., n.$

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5. Let X_1, X_2 be a random sample from the $N(\theta, 1)$ population, define $\phi(X_1) = E[\bar{X}|X_1]$, then $V(\phi(X_1)$ is

(a) $\frac{1}{6}$.
(b) $\frac{1}{4}$.
(c) $\frac{1}{2}$.
(d) $\frac{2}{3}$.

6. Let $X_1, ..., X_n$ be a random sample from a distribution whose parameter θ is greater than -1 and pdf

$$f(x;\theta) = \begin{cases} (1+\theta)x^{\theta} & 0 < x < 1 \\ 0 & otherwise \end{cases}$$

Then MLE of θ is

(a) $-\left[n(log\prod_{i=1}^{n} X_i)^{-1} + 1\right]$.
(b) $\frac{n}{\sum_{i=1}^{n} logX_i} + 1$.
(c) $1 - \frac{n}{\sum_{i=1}^{n} logX_i}$.
(d) $\left[n(log\prod_{i=1}^{n} X_i)^{-1} + 1\right]$.

7. Let X be a non-negative continuous random variable with finite mean μ. Let $h(.)$ be the hazard rate of X, then

(a) $E(h(X)) = \frac{1}{\mu}$.
(b) $E(h(X)) = \mu$.
(c) $E(h(X)) \geq \frac{1}{\mu}$.
(d) $E(h(X)) < \frac{1}{\mu}$.

8. $\{N(t), t \geq 0\}$ is Poisson process with parameter $\frac{1}{2}$ and let Y_k be time till the k^{th} arrival, $P(Y_k > 2)$ is equal to

(a) $e^{-1}\sum_{j=0}^{k-1} \frac{1}{j!}$.
(b) $e^{-1}\sum_{j=0}^{k} \frac{1}{j!}$.
(c) e^{-1}.
(d) e^{-2}.

9. Consider the function $f(x_1, x_2) = x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_2 + 2x_1, \quad x_1, x_2 \in \mathbb{R}$, let $H(x_1, x_2)$ denotes its Hessian, which of the following is correct?

   (a) $H(x_1, x_2)$ positive definite and hence $f(x_1, x_2)$ is convex.

   (b) $H(x_1, x_2)$ positive definite but nothing can be said about the convexity of the function.

   (c) $H(x_1, x_2)$ indefinite.

   (d) $H(x_1, x_2)$ negative definite.

10. $X \sim Poisson(2)$, $Y \sim Poisson(3)$ and are independent. If $X + Y = 10$, the variance of X is

   (a) $\frac{12}{5}$.

   (b) $\frac{6}{5}$.

   (c) $\frac{12}{25}$.

   (d) $\frac{6}{25}$.

11. X and Y are two random variables where $P(X > Y) = 1$, then which of the following is correct?

   (a) $E(X) \geq E(Y)$.

   (b) $E(X) > E(Y)$.

   (c) $E(X) < E(Y)$.

   (d) Nothing definite can be said.

12. For a random variable X with parameter θ, if the functions $L(.)$ and $U(.)$ satisfy $P_\theta\left(L(X) \leq \theta\right) = 1 - \alpha_1$ and $P_\theta\left(L(X) \leq \theta \leq U(X)\right) = 1 - \alpha_1 - \alpha_2$ and $L(x) \leq U(x)$ for all x then $P_\theta\left(U(X) \geq \theta\right)$ is

   (a) $1 - \alpha_2$.

   (b) α_2.

   (c) $\frac{\alpha_1 + \alpha_2}{2}$.

   (d) $\frac{\alpha_1 \alpha_2}{2}$.

13. In a randomized block design of 4 treatments and 3 blocks the degrees of freedom of the residual sum of squares(Error sum of squares) is

   (a) 3.

   (b) 4.

   (c) 5.

   (d) 6.

14. A population of 30 units is divided into three strata with 6, 12,12 units each. The number of different ways in which a stratified random sample of 5 units can be drawn in accordance to proportional allocation is

(a) $6^3 \times 11^2$.
(b) $5 \times 6^3 \times 11$.
(c) 20×12^2.
(d) $25 \times 11 \times 12$.

15. For any Gauss Markov model $(\mathbf{Y}_{n\times 1}, \mathbf{X}\beta_{p\times 1}, \sigma^2\mathbf{I})$, with rank of $\mathbf{X}$ equal to $p-1$ and $n > p$, which of the following is correct?

(a) Every component of $\beta_{p\times 1}$ is certainly estimable.
(b) Certainly no component of $\beta_{\mathbf{p}\times\mathbf{1}}$ is estimable.
(c) Every component of $\beta_{\mathbf{p}\times\mathbf{1}}$ may be non estimable.
(d) Exactly one component of $\beta_{\mathbf{p}\times\mathbf{1}}$ is estimable.

16. In a hypothesis testing problem, the $p-$value was 0.06, which of the following is a correct decision?

(a) The null hypothesis should be rejected at 0.05 level of significance.
(b) The null hypothesis should be accepted at 0.05 level of significance.
(c) The null hypothesis should be accepted at 0.07 level of significance.
(d) None of the above.

17. Customers arrive in a super market in accordance with a homogeneous Poisson process, if the expected number of arrivals in one hour is 20, the expected length of time between the arrival times of the 6^{th} and 7^{th} customer is

(a) 3 minutes.
(b) 10 minutes.
(c) 5 minutes.
(d) 30 minutes.

18. $\mathbf{X} \sim N_{20}(\mathbf{0}, \mathbf{\Sigma})$ where $\mathbf{X}' = (X_1, X_2, \ldots, X_{20})$, the diagonal elements of $\mathbf{\Sigma}$ are 1 and the off diagonal elements are $\frac{1}{2}$, then $Y_1 = \sum_{i=1}^{20} X_i$ and $Y_2 = \sum_{i=1}^{20} a_i X_i$ are independently distributed if

(a) $\sum_{i=1}^{20} a_i = -20$.
(b) $\sum_{i=1}^{20} a_i = 0$.
(c) $\sum_{i=1}^{20} a_i = -1$.
(d) such a Y_2 cannot be determined.

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19. X is a random variable with probability distribution

$$P(X = +1) = P(X = -1) = \frac{1}{2}$$

its characteristic function is

(a) $\frac{t}{2}$.
(b) e^{it}.
(c) sint.
(d) cost.

20. Which of the following is always correct for any 5×5 real, skew-symmetric matrix A?

(a) det $A > 0$.
(b) det $A > 0$.
(c) A is singular.
(d) A is definite.

21. $X_i \sim U(-1, +1), i = 1, 2, \ldots$ and are independent. $Lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i > 0\right)$

(a) is 0.
(b) is $\frac{1}{2}$.
(c) is 1.
(d) does not exist.

22. $X_i, i = 1, 2, 3, \ldots$ are independently distributed with the following distributions

$$P(X_i = -1) = P(X_i = 0) = P(X_i = +1) = \tfrac{1}{3} \quad for\ i = 1, 3, 5, \ldots$$
$$P(X_i = -1) = P(X_i = +1) = \tfrac{1}{2} \quad for\ i = 2, 4, 6, \ldots$$

Let $Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then

(a) $Y_n \to 1$ almost surely.
(b) $Y_n \to 0$ almost surely.
(c) $Y_n \to 0$ in probability but not almost surely.
(d) $Y_n \to 1$ in probability but not almost surely.

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23. A fair die is rolled and then a fair coin is tossed as many times as the number that shows up on the die, the expected number of heads is

(a) 4.
(b) 3.
(c) $\frac{7}{2}$.
(d) $\frac{7}{4}$.

24. The transition probability matrix of a Markov chain with state space $S = \{0, 1, 2\}$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Which of the following statements is not correct?

(a) This Markov chain is irreducible.
(b) This Markov chain has a stationary distribution.
(c) This Markov chain is recurrent.
(d) This Markov chain is aperiodic.

25. The multiple correlation coefficient between Y and X_1, X_2, X_3 is 0.98, and between Y and X_1, X_2 is 0.91. The partial correlation coefficient between Y and X_3 after removing the effects of X_1 and X_2 is in the interval

(a) (0.3,0.4].
(b) (0.5,0.7].
(c) (0.7,0.8].
(d) (0.8,0.9].

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PART B

- There are 15 questions in this part. Answer as many as you can.
- The maximum you can score is 50. Marks are indicated against each question.
- The answers should be written in the separate answer script provided to you.

1. An urn contains 10 balls of which X are red and the rest are white, where X takes values $0, 1, 2, 3$ with probabilities $\frac{1}{4}$ each. 3 balls were drawn without replacement of which 2 were white and 1 was red, determine the probabilities that $0, 1, 2, 3$ balls in the urn are red. **[5 marks]**

2. The conditional density function of X given $Y = y$ is

$$f(x|y) = \begin{cases} (1+y)x^y & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

and the marginal distribution of Y is $U(-1, 1)$, determine $E(X)$. **[4 marks]**

3. Candidates are allowed to appear for the civil service exams at most 4 times with the condition that he cannot write the exam if he has cleared in any of the earlier attempts. . For Ashok the probabilities of the clearing the exam in the first attempt is 0.3, in the second it is 0.5, in the third it is 0.6 and in the fourth it is 0.8.

   (a) Determine the probability of failing in the 1^{st}, 2^{nd}, 3^{rd} and the 4^{th} attempts.

   (b) Expected number of attempts. **[6 marks]**

4. Let X and Y be independent standard normal random variables and let R and Θ be the polar coordinates of the vector (X, Y). Show R^2 and Θ are independent with R^2 being exponential with mean 2 and Θ being uniformly distributed over $(0, 2\pi)$. **[5 marks]**

5. A random variable Y is said to have Weibull distribution with parameter $\alpha, \beta > 0$ if its distribution function is given by

$$F(y) = \begin{cases} 0 & y < 0 \\ 1 - e^{-(\frac{y}{\alpha})^\beta} & y \geq 0 \end{cases}$$

Define $X = (\frac{y}{\alpha})^\beta$, what is the distribution of X? **[4 marks]**

6. Suppose that $X_1, \ldots, X_n$ is a random sample from a distribution with pdf

$$f(x;\theta) = \begin{cases} \theta^2 x e^{-\theta x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

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where $\theta > 0$. Find the UMVUE of θ, using Rao-Blackwell-Lehmann-Scheffe theorem. **[4 marks]**

7. $\{X_n\}_1^\infty$ is a sequence of independent random variable with following distribution

$$P(X_n = -1) = P(X_n = +1) = \frac{1}{n^2+1}$$

and

$$P(X_n = 0) = \frac{n^2-1}{n^2+1}$$

Show that $P\left(\frac{S_n}{\sigma_n} \leq z\right) \rightarrow \Phi(z)$ as $n \rightarrow \infty$, where $S_n = X_1 + ... + X_n$ and $\sigma_n^2 = V(X_1 + ... + X_n)$. **[6 marks]**

8. Let $X_1, ..., X_n$ be a random sample from the $N(\theta, \theta)$, $\theta > 0$ population. Give an example of a Pivotal Quantity and use it to obtain a $(1-\alpha)100\%$ confidence interval of θ. **[5 marks]**

9. Show that the Fisher information contained in a sample of size 4 from a Cauchy distribution with location parameter θ is 2. **[3 marks]**

10. $\mathbf{X}' = (X_1, X_2, X_3, X_4)$ and $\mathbf{X} \sim N(\mathbf{0}, \frac{1}{2}(I+J))$, where J is a 4×4 matrix in which every element is 1.

    (a) Compute $E\left(\binom{X_3}{X_4} \middle| \binom{X_1}{X_2} = \binom{-2}{3}\right)$ and $D\left(\binom{X_3}{X_4} \middle| \binom{X_1}{X_2} = \binom{-2}{3}\right)$ **[5 marks]**

    (b) Show that $\frac{4}{3}(X_1^2 + X_2^2 - X_1X_2) \sim \chi^2(1)$ **[4 marks]**

11. A population consists of N units $U_1, ..., U_N$. X and Y are two variables of interest. Obtain an unbiased estimator for population covariance

$$\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})\,(Y_i - \bar{Y})$$

based on a srswor of size n. (X_i, Y_i) are values of (X, Y) respectively for U_i, $i = 1, 2, ..., N$. **[5 marks]**

12. The starting and current simplex tableaus of a given linear programming problem(minimization problem) are given below. Find the values of unknowns **a,b,...,l** **[6 marks]**

Starting Tableaux

	z	x_1	x_2	x_3	x_4	x_5	RHS
	1	a	1	-3	0	0	0
x_4	0	b	c	d	1	0	6
x_5	0	-1	2	e	0	1	1

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Current Tableaux

	z	x_1	x_2	x_3	x_4	x_5	RHS
	1	0	$\frac{-1}{3}$	j	k	l	-4
x_1	0	g	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	f
x_5	0	h	i	$\frac{-1}{3}$	$\frac{1}{3}$	1	3

13. Consider the problem

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} x_j \\ \text{subject to} \quad & \prod_{j=1}^{n} x_j = 1 \\ & x_j \geq 0, \quad j = 1, \ldots, n \end{aligned}$$

(a) What are the KKT conditions for the problem?

(b) Using the KKT conditions, find an optimal solution to the problem.

[5 marks]

14. Let X_n be the size of a group that enters a restaurant at time n, $n = 1, 2, \ldots$, further $X_n's$ are i.i.d random variables taking values $1, 2, \ldots, d$ with probabilities $\frac{1}{d}$ each. Now define a random variable Y_n to be the size of the largest group that entered the restaurant till time n.

(a) Show that the $\{Y_n, n = 1, 2, \ldots\}$ is a Markov chain and write down its transition probability matrix.

(b) Classify the states into communicating classes and identify the recurrent and transient classes.

(c) For each transient state j and for each recurrent state k compute f_{jk}.

[10 marks]

15. A sample of n units is selected from a population of N units without replacement in the following way. The first selection is made with unequal probabilities while the remaining $n-1$ units are selected with equal probabilities. Let the first draw selection probabilities be $P_1, P_2, \ldots, P_N$ for the population units $U_1, U_2, \ldots, U_N$ respectively. Define

$$\delta_i = \begin{cases} 1 & \text{if } U_i \text{ is included in the sample} \\ 0 & \text{otherwise} \end{cases}$$

Find Π_i(Probability that U_i is in the sample) and Π_{ij}(Probability that U_i and U_j are in the sample) for each $i, j = 1, 2, \ldots, N$. **[6 marks]**