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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, August 2009
Branch : Mathematics
MM 221 : ALGEBRA
(Prior to 2005 Admn.)

Time: 3 Hours Max. Marks: 75

Instructions : *1) Answer* ***5*** *questions choosing Part –* ***A*** *or Part –* ***B*** *from* ***each*** *question.*
2) ***All*** *questions carry* ***equal*** *marks.*

1. A) a) Prove that $Z_m \times Z_n \simeq Z_{mn}$ if m and n are relatively prime integers. What can be said about $Z_2 \times Z_2$?

b) Derive the conditions which are necessary and sufficient for a group G to be the internal direct product of its subgroups H and K.

B) a) Show that if m divides the order of a finite abelian group then G has a subgroup of order m.

b) Find, upto isomorphism, all abelian groups of order 60.

c) Show that if G has a composition series and if N is a normal subgroup of G, then G has a composition series.

2. A) a) Let X be a G-set for a group G. Show that $G_x = \{g \in G \mid xg = x\}$ is a subgroup of G for each $x \in X$.

b) Show that if X is a G-set for a group G, the relation $x_1 \sim x_2$ if and only if $x_1 g = x_2$ for some $g \in G$, is an equivalence relation on X.

c) Show that every group of order p^2 is abelian.

P.T.O.

B) a) Show that if H is a p-subgroup of a finite group G and N [H] is the normaliser of H in G, then $(N[H]:H) \equiv (G:H) \pmod{p}$.

b) Derive the class equation for a finite group G.

c) Show that a group of order 15 has a normal subgroup.

3. A) a) Show that if A is a $n \times n$ matrix in F the function $\langle X, Y \rangle = X^t AY$ defined on the space F^n of column vectors is a bilinear form and it is symmetric if and only if A is symmetric.

b) If P is an element in SU_2 with eigen values λ and $\bar{\lambda}$, show that P is conjugate in SU_2 to the matrix $\begin{bmatrix} \lambda & . \\ . & \lambda \end{bmatrix}$.

c) Show that if A is a skew symmetic matrix, then e^A is orthogonal.

B) a) Show that if V is a m-dimensional vector space over a field of characteristic $\neq 2$ and $\langle , \rangle$ is a nondegenerate skew-symmetric form on V, then the dimension m of V is even.

b) Show that SU_2 is homeomorphic to the unit 3 sphere in $\mathbb{R}^4$.

4. A) a) Show that if E is a finite extension of F and K is a finite extension of E, then K is a finite extension of F and $[K:F] = [K:E][E:F]$.

b) Find the degree $Q\left(\sqrt{2}+\sqrt{3}\right)$ over Q.

c) Show that squaring the circle is impossible.

B) a) Let E be a field and F a subfield of E. Show that the set G (E / F) of all automorphisms of E leaving F fixed forms a subgroup of the group of all automorphisms of E and $F \leq E_{G(E/F)}$.

b) If F is a finite field of characteristic p, show that the map $\sigma_p : F \to F$ defined by $a\sigma_p = a^p$ is an automorphism of F and $F_{\sigma_p} \simeq Z_p$.

c) Let E be finite extension of F and σ an isomorphism of F onto a field F' and let $\overline{F'}$ be an algebraic closure of F. Show that the number of extensions of σ to an isomorphism τ of E into $\overline{F'}$ is finite.

5. A) a) If E is a field such that $F \leq E \leq \overline{F}$, show that E is a splitting field over F if and only if every automorphism of $\overline{F}$ leaving F fixed maps E onto itself.

b) Find the splitting field of $X^3 - 2$ over Q and its degree.

B) a) If E is a finite extension of F and K is a finite extension of E, show that K is a separable extension of F if and only K is a separable extension of E and E is a separable extension of F.

b) If F is a finite field containing q elements and E is a finite extension of degree n over F show that E contains q^n elements.

c) Is $\mathbb{R}$ a splitting field of Q ? Is $\mathbb{C}$ a splitting field of $\mathbb{R}$?