Reg. No..................................

(Pages : 2)

K 5103

Name..................................

FINAL YEAR B.Sc. DEGREE EXAMINATION, MARCH/APRIL 2005

Part III—Group I—Mathematics

Paper III—ALGEBRA

Time : Three Hours

Maximum : 65 Marks

Maximum marks for each unit is 13.

Unit I

1. Define a relation $\mathcal{R}$ on z by setting $n\mathcal{R}m$ if and only if $nm \geq 0$. Verify whether $\mathcal{R}$ is an equivalence relation on z.

(3 marks)

2. Define $*$ on Q^+ by $a * b = \frac{ab}{2}$. Prove that $(Q^+, *)$ is a group. (4 marks)

3. Let $a \in G$, G a group. Prove that $H = \{a^n : n \in z\}$ is a subgroup of G and that it is the smallest subgroup of G that contains a.

(4 marks)

4. Prove that every group is isomorphic to a group of permutations. (5 marks)

5. Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$ as a product of transpositions. (4 marks)

Unit II

6. Show that every group of prime order is cyclic. (5 marks)

7. Compute the factor group $z_4 \times z_6/\langle(0, 1)\rangle$. Verify whether it is isomorphic to z_4.

(5 marks)

8. If H is a subgroup of G and N is a normal subgroup of G show that $\frac{HN}{N} \simeq H/(H \cap N)$.

(5 marks)

9. Let X be a G-set. Define the isotropy subgroup G_x of $x \in X$. Prove that G_x is a subgroup of G for each $x \in X$.

(5 marks)

Turn over

Unit III

10. Prove that z_n under addition modulo n and multiplication modulo n form a ring. Verify whether it is an integral domain.

(5 marks)

11. Define the characteristic of a ring R. Prove that if R is a ring with unity then R has characteristic $n > 0$ if and only if n is the smallest positive integer such that $n \cdot 1 = 0$.

(5 marks)

12. Show that the quarternions form a skew field under addition and multiplication.

(5 marks)

13. Prove that any *two* fields of quotients of an integral domain are isomorphic. (5 marks)

Unit IV

14. Let A, B be ideals of a ring R. Define

$$A + B = \{a + b / a \in A, b \in B\}.$$

Show that A + B is an ideal of R.

(5 marks)

15. Define a prime ideal of a ring. Prove that an ideal $N \neq R$ is prime if and only if $\frac{R}{N}$ is an integral domain where R is a commutative ring with unity.

(5 marks)

16. Prove that a non-zero polynomial $f(x) \in F[x]$ of degree n can have at most n zeroes in a field F.

(5 marks)

17. If F is a field prove that every non-constant polynomial $f(x) \in F[x]$ can be expressed uniquely as a product of irreducible polynomials.

(5 marks)

Unit V

18. Let V be a vector space over F and F^1 be a subfield of F. Show that V is a vector space over F^1 also.

(5 marks)

19. Show that if V_1 is a subspace of V_2 and V_2 is a subspace of V, then V_1 is a subspace of V.

(5 marks)

20. Prove that

$$\dim (V_1 + V_2) = \dim V_1 + \dim V_2 - \dim (V_1 \cap V_2),$$

where V_1 and V_2 are subspaces of a vector space V.

(5 marks)

21. Prove that if V, V^1 are vector spaces over a field F then the set of all linear transformations of V to V^1 form a vector space over F.

(5 marks)