Reg. No.................................. (Pages : 3) **K 5182**

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SIXTH SEMESTER B.C.A. DEGREE EXAMINATION, FEBRUARY/MARCH 2005

(Vocational Course)

Optional Subject : Mathematics

Paper XI—REAL AND COMPLEX ANALYSIS

Time : Three Hours Maximum : 90 Marks

Section A

(*Maximum* : 40 *marks*)

Each question carries 5 *marks.*

1. If A is closed and G is open, prove that

   (a) G—A is open.

   (b) A—G is closed.

2. Show that the only limit point of $S = \left\{a + \frac{1}{n} : n \in N\right\}$ is a.

3. Prove that countable union of countable sets is countable.

4. Given $a_1 > 0$ and $a_{n+1} = \frac{1}{a_n} + \frac{a_n}{2} \; \forall \; n \in N$. Show that $\langle a_n \rangle$ converges to $\sqrt{2}$.

5. Prove that (0, 1] is uncountable.

6. State and prove Cauchy's first theorem on limits.

7. Examine the convergence of the series :

   (a) $\sum \left(n^2 (n+1)\right)^{-1/2}$.

   (b) $\sum \left\{\left(n^3 + 1\right)^{1/3} - n\right\}$.

8. Show that the function $f(x) = \sin x^2$ is continuous and bounded on R, but not uniformly continuous on R.

9. Show that the series $\sum \frac{(-1)^{n-1}}{x^2 + n}$ converges uniformly on R but not absolutely.

Turn over

10. Define limit point of a sequence. Find the limit superior and limit inferior of the following :—

    (a) sequence $\langle a_n \rangle$ where $a_n = \sin \frac{n\pi}{3}, n \in \mathrm{N}$.

    (b) sequence $\langle a_n \rangle$ where $a_n = \frac{(-1)^n}{n}, n \in \mathrm{N}$.

11. Show that the exponential function E satisfies $E(x+y) = E(x)\,E(y)$ for all $x, y \in \mathrm{R}$.

12. State and prove Weierstrass M-test.

(8 × 5 = 40 marks)

Section B

(*Maximum* : 40 *marks*)

Each question carries 5 marks.

13. Show that the function $f(z) = |z|^2$ is differentiable at the origin, but not analytic there.

14. Find the equation of the circle described on the line joining $1 + i$ and $1 - i$ as diameter.

15. If a function is analytic, prove that it is independent of $\bar{z}$.

16. State and prove Liouville's theorem.

17. State and prove Cauchy's Integral formulae.

18. Expand $\frac{z-1}{z^2}$ about $z = 1$ in :

    (a) Taylor's series.

    (b) Laurent's series.

19. State and prove Cauchy's residue theorem.

20. Using contour integration along the unit circle, show that $\int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}, a > |b|$.

21. Using contour integration, evaluate

$$\int_0^{\infty} \frac{1}{\left(x^2+1\right)^2}\, dx.$$

22. Find the bilinear transformations which maps the points $-i, o, i$ into $-1, i, 1$ respectively.

23. Show that both the transformations $w = \frac{1+z}{1-z}$, $w = \frac{z+1}{z-1}$ map the left half plane $\mathrm{Re}(z) \le 0$ onto $|w| \le 1$.

24. Discuss the transformation $w = \sqrt{z}$.

(8 × 5 = 40 marks)

Section C

Answer **all** *the five questions.*
Each question carries 2 marks.

25. If $a > 0$, show that $\lim\limits_{n \to \infty} \dfrac{n}{(1+a)^n} = 0.$

26. Define interior of a set and prove that it is always open.

27. Show that $z = 0$ is an essential singularity of the function $\sin\left(\dfrac{1}{z}\right)$.

28. If $f(z)$ and $\overline{f(z)}$ are analytic in a region, show that $f(z)$ is constant in that region.

29. Find an analytic function with real part $2xy$.

(5 × 2 = 10 marks)