Math 1.2

I Semester M.Sc. Examination, May 2011
MATHEMATICS
Real Analysis – I

Time : 3 Hours Max. Marks : 80

Note : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Prove the following statements in the field of real numbers : **(2+1+2+1)**
   i) If x + y = x + z , then y = z
   ii) If x + y = x, then y = 0
   iii) If x + y = 0, then y = –x
   iv) – (–x) = x.

   b) Prove that Q is dense in R and R/Q is also dense in R. **(5 + 5)**

2. a) Define a countable set. Show that the set $\mathbb{Z}$ of all integers is countable. Further, show that every subset of a countable set is also countable. **8**

   b) Define a limit point and closed set in a metric space. Show that every point of an open interval (a, b) in R' is a limit point. Further, show that the closed interval (a, b) is a closed set in R'. **8**

3. a) In a metric space show that closed subsets of a compact set are also compact. **5**

   b) Prove that the closed interval (a, b) is compact in R'. **6**

   c) Prove that a subset E of R' is connected if and only if it is an interval. **5**

4. a) Prove that limit of a convergent sequence is unique. **4**

   b) If $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n \to \infty} y_n = y$, then show that $\lim\limits_{n \to \infty} x_n\, y_n = xy$. **4**

   c) Define $x_1 = 1$, $x_n + 1 = \dfrac{4+3x_n}{3+2\,x_n}$, $n \geq 1$. **8**

   Find x_2, x_3 and x_4. Show that $\{x_n\}$ is monotonically increasing and bounded sequence. Compute the limit of the sequence.

P.T.O.

5. a) Define the number e. Derive $\lim_{n \to \infty} \left(1+\frac{1}{n}\right)^n = e$ **6**

b) Prove that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. **6**

c) Show that if $x_n = \frac{n^n}{n!}$, then $\lim_{n \to \infty} x_n = e^n$. **4**

6. a) Prove that $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converses if p > 1 and diverses if $p \le 1$. **8**

b) Investigate the behavior of the following series :

i) $\sum_{n=1}^{\infty} \left(\sqrt[n]{n}-1\right)^n$

ii) $\sum_{n=0}^{\infty} \left(\frac{n^3+1}{2^n+1}\right)$ **(4+4)**

7. a) Prove that if $\Sigma\ a_n$ converges and if $\{b_n\}$ is monotonic and bounded, then $\Sigma\ a_n b_n$ converges. **5**

b) State Kummer's test for convergence or divergence of a series. Deduce Ratio test and Raabe's test from Kummer's test. **6**

c) Discuss the convergence or divergence of $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$. **5**

8. a) State and prove Merten's theorem for Cauchy product of two series. **10**

b) If $\Sigma\ a_n$ is a series of complex numbers which converges absolutely, then prove that every rearrangement of $\Sigma\ a_n$ converges. **6**