Indian Institute of Technology Kharagpur

Departments: IM, NA and EC.

MA20105 Linear Algebra

Autumn End Semester Examination, 2010 No. of Students: 100

Full Marks: 50, Time: 3 Hrs.

INSTRUCTION: Answer any 10 questions. Each question carries equal marks.

1. (a) Is $V = \mathbb{R}^2$ a vector space over $\mathbb{R}$ with respect to the operations: $(a,b)+(c,d) = (a+c, b+d)$ and $k(a,b) = (k^2a, k^2b)$?

   (b) Let V be a vector space of dimension n over a field F. Then show that any subset of n vectors of V that generates V is a basis of V.

   (2+3 = 5 marks)

2. (a) Find the dimension of the vector space U of $n \times n$ symmetric matrices over a field F?

   (b) Let V be the vector space of all real polynomials $P(x)$ and $T : V \to V$ is defined by $T(P(x)) = xP(x), P(x) \in V$, $D : V \to V$ is defined by $D(P(x)) = \frac{d}{dx}P(x), P(x) \in V$. Describe the mappings TD and DT. Are they equal?

   (2+3 = 5 marks)

3. (a) Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the basis vectors $(0,1,1), (1,0,1), (1,1,0)$ of $\mathbb{R}^3$ to $(1,1,1), (1,1,1), (1,1,1)$ respectively. Verify that $\dim\ker(T) + \dim \text{Range}(T) = 3$.

   (b) Prove that each eigen value of a real orthogonal matrix has unit modulus.

   (3+2 = 5 marks)

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4. (a) Let U and V be vector spaces over the field F and let T be a linear transformation from U into V. If T is one-one and onto, then show that the inverse function T^{-1} is a linear transformation from V into U.

   (b) The set $\{1, t, e^t, te^t\}$ is a basis of a vector space V of functions $f : \mathbb{R} \to \mathbb{R}$. Let D be the differential operator on V, that is, $D(f) = \frac{df}{dt}$. Find the matrix of D in the given basis.

(2+3= 5 marks)

5. (a) Prove that two finite dimensional vector spaces V and W over a field F are isomorphic if and only if $\dim V = \dim W$.

   (b) Let V be an inner product space. Show that

   (i) $\langle \theta, \beta \rangle = 0, \ \forall \beta$ in V.

   (ii) If $\langle \alpha, \beta \rangle = 0, \ \forall \beta$ in V, then $\alpha = \theta$.

(3+2= 5 marks)

6. State and prove the Cayley-Hamilton theorem for an $n \times n$ matrix A.

(5 marks)

7. (a) If W_1 and W_2 are subspaces of the vector space $V(F)$, then prove that

   (i) $W_1 + W_2$ is a subspace of $V(F)$.

   (ii) $L(W_1 \cup W_2) = W_1 + W_2$, where $L(W_1 \cup W_2)$ is the linear span of $(W_1 \cup W_2)$.

   (b) If $c \in F$ is an eigen value of a linear operator T on a vector space $V(F)$, then show that for any polynomial $P(x)$ over F, $P(c)$ is an eigen value of $P(T)$.

(3+2= 5 marks)

8. Find whether the matrix $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$ is diagonalizable. If so, find the matrix P^{-1} and show that $P^{-1}AP = D$.

(5 marks)

P. T. O

9. (a) Prove that if α and β are vectors in a Unitary space, then
   (i) $4\langle\alpha,\beta\rangle = ||\alpha+\beta||^2 - ||\alpha-\beta||^2 + i||\alpha+i\beta||^2 - i||\alpha-i\beta||^2$.
   (ii) $\langle\alpha,\beta\rangle = \text{Re}\,\langle\alpha,\beta\rangle + i\text{Re}\,\langle\alpha,i\beta\rangle$.

   (b) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z) = (x-y, x+2y, y+3z)$, $\forall(x,y,z) \in \mathbb{R}^3$. Show that T is non-singular and determine T^{-1}.

(3+2= 5 marks)

10. (a) State and prove the Cauchy-Schwarz's inequality for an inner product space V.

    (b) Give definition of an orthonormal set with an example.

(4+1= 5 marks)

11. (a) State the Gram-Schmidt orthogonalization process. If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is an orthonormal set in V and if $\beta \in V$ then show that $\gamma = \beta - \sum_{i=1}^{m}\langle\beta,\alpha_i\rangle\alpha_i$ is orthogonal to each of $\alpha_1, \alpha_2, \ldots, \alpha_m$ and, consequently to the subspace spanned by S.

    (b) Let V be the vector space of all functions from the real field $\mathbb{R}$ into $\mathbb{R}$. Let U be the subspace of even functions and W the subspace of odd functions. Show that $V = U \oplus W$.

(3+2= 5 marks)

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