Indian Institute of Technology Kharagpur

Department: EC, MI and IE.

MA20107 Matrix Algebra

Autumn End Semester Examination, 2010 No. of Students: 100

Full Marks: 50, Time: 3 Hrs.

INSTRUCTION: Answer any 10 questions. Each question carries equal marks.

1. (a) If U_1, U_2, U_3 are subspaces of a finite dimensional vector space, then $\dim(U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3)$. Prove this or give a counter example.

   (b) If λ is an eigen value of a real orthogonal matrix A, prove that $\frac{1}{\lambda}$ is also an eigen value of A.

   (2+3 = 5 marks)

2. (a) Let V be a vector space over $\mathbb{R}$ and $S \subset V$ a subset (not necessarily a subspace). Show that the following two conditions on S are equivalent:

   (i) S is non-empty; and if $x, y \in S$ and $\lambda \in \mathbb{R}$, then $\lambda x + (1-\lambda)y \in S$.

   (ii) There is a vector $v \in V$ and a subspace W of V, such that $x \in S \Leftrightarrow x - v \in W$. (Such an S is called an *affine subspace* of V).

   (b) If S, T are subsets of $V(F)$, then show that $S \subseteq L(T) \Rightarrow L(S) \subseteq L(T)$, where $L(S), L(T)$ denote the linear span of S and T respectively.

   (4+1 = 5 marks)

3. (a) Let V and W be finite dimensional vector spaces of the same dimension over a field F and $T : V \to W$ be a linear mapping. Then show that T is one-to-one $\Leftrightarrow$ T is onto.

   (b) If α, β are vectors in an inner product space $V(F)$ and $a, b \in F$, then prove that

   (i) $||a\alpha + b\beta||^2 = |a|^2||\alpha||^2 + a\bar{b}\langle\alpha, \beta\rangle + \bar{a}b\langle\beta, \alpha\rangle + |b|^2||\beta||^2$.

   (ii) $\mathrm{Re}\langle\alpha, \beta\rangle = \frac{1}{4}||\alpha + \beta||^2 - \frac{1}{4}||\alpha - \beta||^2$.

   (2+3 = 5 marks)

P. T. O

4. (a) Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_1, x_1 - x_3).$$

Find the matrix representation of T with respect to the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$, where $\alpha_1 = (1,0,1), \alpha_2 = (0,1,1), \alpha_3 = (1,1,0)$.

(b) Determine whether $\ln(I+A) = \sum_{n=0}^{\infty} \left[\frac{(-1)^{n+1}}{n}\right] A^n$ is well-defined for the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & -1 & -3 \end{pmatrix}$.

(3+2 = 5 marks)

5. (a) If λ be an eigen value of a matrix A of order $n \times n$ and W_λ be the set of eigen vectors corresponding to eigen value λ together with $\{\theta\}$, that is, $W_\lambda = \{X | AX = \lambda X\}$, then show that the set W_λ is a subspace of $V_n(F)$.

(b) Let $M = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$, where A and B are square matrices. Show that the minimal polynomial $m(t)$ of M is the least common multiple of the minimal polynomials $g(t)$ and $h(t)$ of A and B respectively.

(2+3 = 5 marks)

6. (a) Prove that the minimal polynomial of a matrix is a divisor of every polynomial that annihilates the matrix.

(b) Examine whether the functions $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$ are linearly independent.

(3+2 = 5 marks)

7. (a) Find two linear operators T and S on $V_2(\mathbb{R})$ such that $TS = \widehat{O}$ but $ST \neq \widehat{O}$.

(b) Find the expression $A^{24} - 3A^{15}$ if $A = \begin{pmatrix} 3 & 2 & 4 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix}$.

(2+3 = 5 marks)

P. T. O

8. (a) Find a generalized eigen vector of type 3 corresponding to the eigen value $\lambda = 7$ for the matrix $A = \begin{pmatrix} 7 & 1 & 2 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{pmatrix}$.

(b) Use Gram-Schmidt process to obtain an orthonormal basis of the subspace of the Euclidean space $\mathbb{R}^4$ with the standard inner product, generated by the linearly independent set $\{(1,1,0,1),\ (1,1,0,0),\ (0,1,0,1)\}$.

(2+3 = 5 marks)

9. (a) Assume that $\lambda = 2$ is the only eigen value for a 5×5 matrix A. Find a matrix in Jordan canonical form similar to A if the complete set of ρ_k numbers associated with this eigen value are as follows: $\rho_5 = \rho_4 = \rho_3 = \rho_2 = \rho_1 = 1$.

(b) Determine a canonical basis for $A = \begin{pmatrix} 4 & 1 & 1 & 2 & 2 \\ -1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$.

(1+4 = 5 marks)

10. (a) Let T be a linear transformation from a vector space U into a vector space V with $\ker T \neq \{\theta\}$. Show that there exist vectors α_1 and α_2 in U such that $\alpha_1 \neq \alpha_2$ and $T\alpha_1 = T\alpha_2$.

(b) Prove that two vectors α and β in a real inner product space V are orthogonal if and only if $||\alpha + \beta||^2 = ||\alpha||^2 + ||\beta||^2$. Does the result hold if V is a complex inner product space? Give reasons for your answer.

(2+3 = 5 marks)

11. (a) Prove that a chain is a linearly independent set of vectors.

(b) Prove Parseval's theorem which states that if $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be an orthonormal basis of a Euclidean space V, then for any vector α in V, we have $||\alpha||^2 = c_1^2 + c_2^2 + \ldots + c_n^2$, where c_i is the scalar component of α along β_i, $i = 1, 2, \ldots, n$.

(3+2 = 5 marks)

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