Department of Mathematics
I I T Kharagpur
MA 20101 Transform Calculus End - Autumn 2010
Max. Marks : 50 Time : 3hrs.
No. of Students : 550

Instructions : (i) Answer ALL the questions. Provide answers to all parts of each question together, otherwise it will be ignored.
(ii) L and L^{-1} denote the Laplace and inverse Laplace transforms, respectively.
(iii)Use $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$ or $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ as Fourier transform of the function $f(t)$.

1.a) State and prove the convolution theorem for Laplace transform. [3 marks]

b) Find $L^{-1}\left\{\frac{1}{1+\sqrt{s}}\right\}$ in terms of error function. [3 marks]

c) Using Laplace transform solve the following differential equations

(i) $ty'' + y' + ty = 0, \quad y(0) = 2, \quad y'(0) = 0.$ [3 marks]

(ii) $y'' + 2y' + 5y = e^{-t}\sin(t), \quad y(0) = 0, \quad y'(0) = 1.$ [4 marks]

2.a) Using Laplace transform solve the following boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \; 0 < x < a, \; t > 0,$$

subject to the boundary conditions: $u(0,t) = 0$ and $u(a,t) = 0$, and initial condition $u(x,0) = \sin\left(\frac{\pi x}{a}\right)$, where $k > 0$ is a constant. [6 marks]

b) Find $f(t)$, if its Fourier sine transform is given by $F_S\{w\} = \frac{e^{-aw}}{w}$ and hence find $F_S^{-1}\left\{\frac{1}{w}\right\}$. [3 marks]

c) Find Fourier cosine transform of the function $f(t)$, if

$$f(t) = \begin{cases} \cos(t), & \text{if } 0 < t < a, \\ 0, & \text{if } t > a. \end{cases}$$ [3marks]

3.a) Using Laplace transform, solve

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \; x > 0, \; t > 0,$$

with $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$, $x > 0$; $u(0,t) = F(t)$, $t > 0$ and the solution $u(x,t)$ remains bounded as $x \to \infty$. [5 marks]

b) Find the Fourier transform of $\exp(-3|t|)$ and hence find the value of the integral $\int_0^\infty \frac{\cos(2t)}{t^2+9}dt$. [3 marks]

c) Find Fourier transform of the function $f(t)$, if

$$f(t) = \begin{cases} e^{-2t}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Hence using convolution theorem for Fourier transform find, $F^{-1}\left\{\frac{1}{(w+2i)^2}\right\}$. [4 marks]

4.a) If the Fourier transform of $f(t)$ defined by $F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$, then show that

(i) Fourier transform of $f(at) = \frac{1}{|a|}F(\frac{w}{a})$.

(ii) Fourier transform of $f(t-a) = e^{iwa}F(w)$.

(iii) Fourier transform of $f(t)\cos(at) = \frac{1}{2}\{F(w-a)+F(w+a)\}$. [3 × 1 = 3 marks]

b) Solve the following boundary value problem in the half-plane $y > 0$ by using Fourier transform

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &+ \frac{\partial^2 u}{\partial y^2} = 0, \ -\infty < x < \infty, \ y > 0; \\ u(x,0) &= f(x), -\infty < x < \infty, \end{aligned}$$

$u(x,y)$ is bounded as $y \to \infty$, and u and $\frac{\partial u}{\partial x}$ both vanishes as $|x| \to \infty$. [6 marks]

c) Show that the following differential equation

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &+ \frac{\partial^2 u}{\partial y^2} = 0, \ -\infty < x < \infty, \ y > 0; \\ u_y(x,0) &= f(x), -\infty < x < \infty, \end{aligned}$$

$u(x,y)$ is bounded as $y \to \infty$, and u and $\frac{\partial u}{\partial x}$ both vanishes as $|x| \to \infty$ can be reduced to

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} &+ \frac{\partial^2 \phi}{\partial y^2} = 0, \ -\infty < x < \infty, \ y > 0; \\ \phi(x,0) &= f(x), -\infty < x < \infty, \end{aligned}$$

where $\phi(x,y) = u_y(x,y)$. Further using the solution of (4b) (or solving independently), show that

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\log[(x-\xi)^2+y^2]d\xi + constant. \qquad \text{[4 marks]}$$

************************GOOD LUCK************************