Mathematical Statistics Paper 2007

IMPORTANT NOTE FOR CANDIDATES

- **Attempt ALL the 25 questions.**
- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-25 (subjective questions) carry twenty one marks each.**
- **Write the answers to the objective questions in the Answer Table for Objective Questions provided on page 7 only.**

1. Let the random variable X have binomial distribution with parameters 3 and θ. A test of hypothesis $H_0 : \theta = 3/4$ against $H_1 : \theta = 1/4$ rejects H_0 if $X \leq 1$. Then the test has

(A) size = 5/32, power = 27/32 (B) size = 5/32, power = 18/32

(C) size = 15/32, power = 27/32 (D) size = 1/32, power = 31/32

2. Let X be a random variable having probability density function

$$f(x; x_0, \alpha) = \begin{cases} \dfrac{\alpha x_0^{\alpha}}{x^{\alpha+1}}, & x > x_0 \\ 0, & x \leq x_0 \end{cases}$$

where $\alpha > 0$, $x_0 > 0$. If $Y = \ln\left(\dfrac{X}{x_0}\right)$, then $P(Y > 3)$ is

(A) $e^{-3\alpha x_0}$ (B) $1 - e^{-3\alpha x_0}$ (C) $e^{-3\alpha}$ (D) $1 - e^{-3\alpha}$

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x, y, z) = (x + y, x - z)$. Then the dimension of the null space of T is

(A) 0 (B) 1 (C) 2 (D) 3

Space for rough work

4. Let $X_1, X_2, \ldots, X_{2n}$ be random variables such that $V(X_i) = 4$, $i = 1, 2, \ldots, 2n$ and Cov $(X_i, X_j) = 3$, $1 \le i \ne j \le 2n$. Then $V(X_1 - X_2 + X_3 - X_4 + \cdots + X_{2n-1} - X_{2n})$ is

(A) n (B) $2n$ (C) $3n - 2$ (D) $n + 1$

5. Let X_1 and X_2 be independent random variables, each having exponential distribution with parameter λ. Then, the conditional distribution of X_1 given $X_1 + X_2 = 1$ is

(A) Exponential with mean 2

(B) Beta with parameters $\lambda/2$ and $\lambda/2$

(C) Uniform on the interval $(0,1)$

(D) Gamma with mean 2λ

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(0, \theta)$. Then the uniformly minimum variance unbiased estimator (UMVUE) of θ is

(A) $\left(\frac{n+1}{n}\right) X_{(n)}$ (B) $X_{(1)} + X_{(n)}$ (C) $2\overline{X}$ (D) $X_{(n)}$

Space for rough work

7. Let A be a 4×4 nonsingular matrix and B be the matrix obtained from A by adding to its third row twice the first row. Then $\det(2A^{-1}B)$ equals

(A) 2 (B) 4 (C) 8 (D) 16

8. Independent trials consisting of rolling a fair die are performed. The probability that 2 appears before 3 or 5 is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{5}$

9. Let $X_1, X_2, \ldots, X_6$ be independent random variables such that

$$P(X_i = -1) = P(X_i = 1) = \frac{1}{2}, \; i = 1,2,3,\ldots,6.$$

Then $P\left[\sum_{i=1}^{6} X_i = 4\right]$ is

(A) $\frac{3}{32}$ (B) $\frac{3}{4}$ (C) $\frac{3}{64}$ (D) $\frac{3}{16}$

10. Let 1, x and x^2 be the solutions of a second order linear non-homogeneous differential equation on $-1 < x < 1$. Then its general solution, involving arbitrary constants C_1 and C_2, can be written as

(A) $C_1(1-x)+C_2(x-x^2)+1$

(B) $C_1x+C_2x^2+1$

(C) $C_1(1+x)+C_2(1+x^2)+1$

(D) $C_1+C_2x+x^2$

11. Let

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & x\neq 0 \\ 0, & x=0. \end{cases}$$

Then

(A) $f'(x)$ is continuous at $x=0$

(B) $f''(x)$ is continuous at $x=0$

(C) $f'(0)$ exists

(D) $f''(0)$ exists

12. Let E and F be two events such that $0 < P(E) < 1$ and $P(E\mid F)+P(E\mid F^c)=1$. Then

(A) E and F are mutually exclusive

(B) $P(E^c\mid F)+P(E^c\mid F^c)=1$

(C) E and F are independent

(D) $P(E\mid F)+P(E^c\mid F^c)=1$

13. Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential distribution with mean $1/\lambda$. The maximum likelihood estimator of the median of the distribution is

(A) $\dfrac{\overline{X}}{(\ln 2)}$ (B) $\overline{X}(\ln 2)$ (C) $\dfrac{(\ln 2)}{\overline{X}}$ (D) $\ln(2\overline{X})$

14. $\lim_{n \to \infty} \dfrac{1 - 2 + 3 - 4 + 5 - 6 + \ldots + (-2n)}{\sqrt{n^2 + 1} + \sqrt{n^2 - 1}}$ equals

(A) ∞ (B) $1/2$ (C) 0 (D) $-1/2$

15. By changing the order of integration, the integral

$$\int_0^1 \int_1^{e^x} f(x, y)\, dy\, dx$$

can be expressed as

(A) $\int_0^1 \int_1^{\ln y} f(x, y)\, dx\, dy$ (B) $\int_0^1 \int_0^{\ln y} f(x, y)\, dx\, dy$

(C) $\int_1^e \int_1^{e^y} f(x, y)\, dx\, dy$ (D) $\int_1^e \int_{\ln y}^1 f(x, y)\, dx\, dy$

16. (a) Let $f(x) = x^3 + 3x - 2, x \in \mathbb{R}$. Show that the equation $f(x) = 0$ has only one real root. Also, find x_0 in the interval $(0,1)$ such that the tangent to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is parallel to the line joining the points $(0, -2)$ and $(1, 2)$. (9)

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with

$$T(1,1) = (0,0,1) \text{ and } T(1,2) = (0,1,1).$$

Then find the linear transformation $T(x, y)$. Also, find the associated matrix referred to the standard bases. (12)

17. (a) Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 2 - x^2$ and the line $y = x$, while the top of the solid is bounded by the plane $z = x + 2$. (9)

(b) Find all the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{\left(n + \frac{1}{n}\right)}$$

converges. (12)

18. The cumulative distribution function of a random variable X is

$$F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x+k}{5}, & k \le x < k+1, \ k = 0,1,2 \\ 1, & x \ge 3. \end{cases}$$

Find :

(a) $P(X = j)$ for all non-negative integers j

(b) $P(X > 2)$

(c) $P(-1 \le X < 1)$.

19. Let $X_1, \ldots, X_n$ be independent random variables with X_k having normal distribution with mean $k\theta$ and variance σ^2 for $k = 1, 2, \ldots, n$. Find the maximum likelihood estimator of θ based on $X_1, \ldots, X_n$. Show that it is an unbiased and consistent estimator of θ. (21)

20. Let the joint probability mass function of random variables X and Y be given by

$$P(X = m, Y = n) = \frac{e^{-1}}{(n-m)!\,m!\,2^n}, \quad m = 0,1,2,\ldots,n; \quad n = 0,1,2,\ldots$$

Find the marginal probability mass functions of X and Y. Also, find the conditional probability mass function of X given $Y = 5$, and that of Y given $X = 6$. (21)

21. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from a distribution having the probability mass function

$$P(X = x) = \theta(1-\theta)^x, \; x = 0,1,2,\ldots$$

where $0 < \theta < 1$. Show that $T = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic. Find the uniformly minimum variance unbiased estimator (UMVUE) of θ. (21)

22. Find the continuous solution of

$$\frac{dy}{dx} + y = g(x),\ 0 \le x < \infty;\ y(0) = 2,$$

where

$$g(x) = \begin{cases} 3, & 0 \le x < \pi/2 \\ \cos x, & x \ge \pi/2. \end{cases}$$

23. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent and identically distributed random variables each with mean 4 and variance 4. Show that for large n,

$$0.5 \leq P\left[16n - 12\sqrt{n} \leq \sum_{i=1}^{n} X_{2i} X_{2i-1} \leq 16n + 12\sqrt{n}\right] \leq 0.9. \qquad (21)$$

24. An urn contains ten balls of which M (an unknown number) are white. To test the hypothesis $H_0 : M = 3$ against $H_1 : M = 7$, three balls are drawn at random from the urn without replacement. If X is the number of white balls drawn, show that the most powerful test rejects H_0 if $X \geq k$, where k is a constant. Find the power, if the size of this test is $11/60$. (21)

25. (a) Evaluate the integral $\iint_R e^{(x^2+y^2)/2}\,dx\,dy$, where R is the region bounded by the lines $y = 0$ and $y = x$, and the arcs of the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. (9)

(b) Let

$$f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Determine whether the function is continuous and differentiable at $(0,0)$. (12)