B.E.
Fifth Semester Examination, 2009-2010
Communication Engineering (EE-301-E)

Note : Attempt any five questions. All questions carry equal marks

Q. 1. (a) Explain and prove convolution theorem in frequency domain and time domain.

Ans. Convolution in Time Domain :

If $$x(t) \xleftrightarrow{ft} X(\omega), \text{ then}$$

$$z(t) = y(t) * x(t) \leftrightarrow Z(\omega) = X(\omega).Y(\omega)$$

Proof :

$$Z(\omega) = \int_{-\alpha}^{\alpha} Z(-1)e^{-j\omega t}.dt$$

$$= \int_{-\alpha}^{\alpha} [x(t) * y(t)]e^{-j\omega t}.dt$$

$$= \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} [x(z)y(t-z)dz]e^{-j\omega t}.dt$$

$$= \int_{-\alpha}^{\alpha} x(z)\left[\int_{-\alpha}^{\alpha} y(t-z)e^{-j\omega t}.dt\right]d\tau$$

Put $t - \tau = \alpha$, then $t = \tau + \alpha$

$dt = d\alpha$, limits will remain same

$$Z(\omega) = \int_{-\alpha}^{\alpha} x(z)\left[\int_{-\alpha}^{\alpha} y(\alpha)e^{-j\omega(\tau+\alpha)}.d\alpha\right].d\tau$$

$$= \int_{-\alpha}^{\alpha} x(z)\left[\int_{-\alpha}^{\alpha} y(\alpha)e^{-j\omega\tau}.e^{-j\omega\alpha}d\alpha\right]d\tau$$

$$= \int_{-\alpha}^{\alpha} x(z)e^{-j\omega\tau}.d\tau \int_{-\alpha}^{\alpha} y(\alpha)e^{-j\omega\alpha}d\alpha$$

$$= X(\omega).Y(\omega)$$

Convolution in Frequency Domain :

$$Z(n) = x[n] * y[n] \xleftrightarrow{DIFT} Z(\Omega) = X(\Omega)Y(\Omega)$$

Now, $$Z(\Omega) = \sum_{n\to\alpha}^{\alpha} Z(n)e^{-j\Omega n}$$

$$\left[\begin{aligned}\text{Putting } Z(n) &= y(n) * x(n) \\ &= \sum_{k\to\alpha}^{\alpha} x(k)y(n-k)\end{aligned}\right]$$

$$Z(\Omega) = \sum_{-\alpha}^{\alpha}\left[\sum_{k=-\alpha}^{\alpha} x(k)\, y(n-k)\right] e^{-j\Omega n}$$

Changing the order of summations :

$$Z(\Omega) = \sum_{k\to-\alpha}^{\alpha} x(k) \sum_{n\to-\alpha}^{\alpha} y(n-k) e^{-j\Omega n}$$

Put $n - k = m$ then the above equation becomes

$$= \sum_{k\to-\alpha}^{\alpha} x(k) \sum_{m\to-\alpha}^{\alpha} y(m) e^{-j\Omega(m+k)}$$

$$= \sum_{K\to-\alpha}^{\alpha} x(k) \sum_{m\to-\alpha}^{\alpha} y(m) e^{-j\Omega m} e^{-j\Omega k}$$

$$= \sum_{K\to-\alpha}^{\alpha} x(k) e^{-j\Omega k} \sum_{m\to-\alpha}^{\alpha} y(m) e^{-j\Omega m}$$

$$Z(\Omega) = X(\Omega) Y(\Omega)$$

Q. 1. (b) Find the Fourier transform and draw the spectrum of a gate function with period $T = \frac{1}{4}$ sec and duration $\tau = \frac{1}{20}$ sec. Assume amplitude is A.

Ans.

$$X(\omega) = \int_{-\alpha}^{\alpha} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-1/2}^{1/2} A.e^{-j\omega t}.dt$$

$$= A\left[\frac{e^{-j\omega t}}{-j\omega}\right]^{1/2}$$

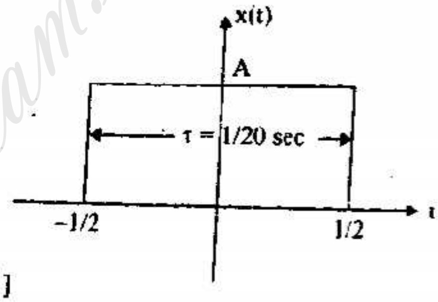
$$= \frac{A}{-j\omega}\left[e^{-j\omega/2} - e^{j\omega/2}\right]$$

$$= \frac{2A}{\omega} \frac{[e^{j\omega/2} - e^{-j\omega/2}]}{2j}$$

$$= \frac{2A}{\omega} \sin\left(\frac{\omega}{2}\right)$$

$$X(\omega) = A \frac{\sin(\omega/2)}{\omega/2}$$

$$X(\omega) = A \sin c\left(\frac{\omega}{2}\right)$$



Magnitude Spectrum :

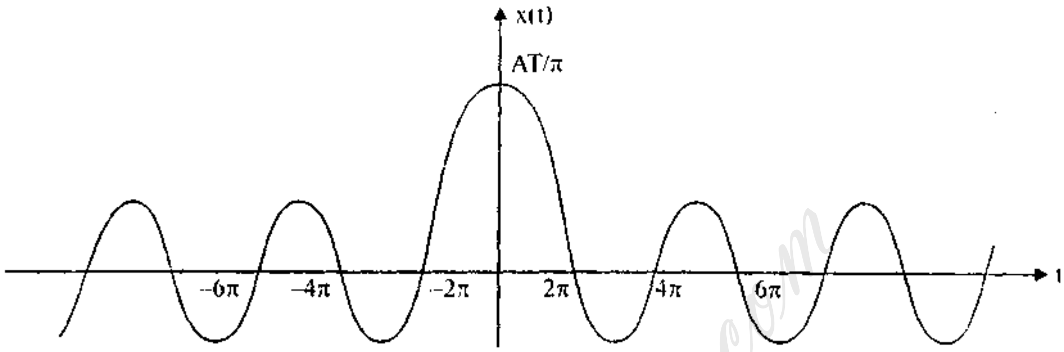
By l's Hospital's rule $\quad |X(\omega)| = A \sin c\left(\frac{\omega}{2}\right)$

$$\angle X(\omega) = 0$$

$$|X(\omega)| = \frac{2A}{\omega}\sin\left(\frac{\omega}{2}\right)$$

Now, this function goes to zero at $\omega = \pm 2n\pi$

Then, $$\lim_{\omega \to 0} \frac{2A}{\omega}\sin\left(\frac{\omega}{2}\right) = \frac{At}{\pi}$$



Q. 2. (a) Find the auto correlation of the following signals (i) sin $(\omega_o t)$, (ii) $e^{-t}u(t)$ and also find power and energy which is applicable.

Ans. (i) $\sin \omega_o t = x(t)$

$$R(\tau) = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)x^*(t-\tau)dt$$

Since the period is not specified

$$R(\tau) = \lim_{T \to \alpha} \frac{1}{T}\int_{-T/2}^{T/2} \sin \omega_o t \sin[\omega_o (t+\tau)]dt$$

$$= \lim_{T \to \alpha} \frac{1}{T}\int_{-T/2}^{T/2} \sin \omega_o t \sin(\omega_o t + \omega_o \tau)dt$$

Since $2\sin x \sin y = \cos(x-y) - \cos(x+y)$

$\therefore$ $$R(\tau) = \lim_{T \to \alpha} \frac{1}{T}\int_{-T/2}^{T/2} [\cos \omega_o \tau - \cos(2\omega_o t + \omega_o \tau)]\,dt$$

$$= \lim_{T \to \alpha} \frac{1}{2T}\cos \omega_o \tau \int_{-T/2}^{T/2} dt - \lim_{T \to \alpha} \frac{1}{2T}\int_{-T/2}^{T/2} \cos(2\omega_c t + \omega_c \tau)dt$$

Since integration of 2nd term will be zero, since its integration of sinusoidal signal over one complete cycle. That is

$$R(\tau) = \lim_{T \to \alpha} \frac{1}{2T}\cos \omega_o \tau [t]_{-T/2}^{T/2}$$

$$R(\tau)=\frac{\cos\omega_o\tau}{2}$$

(ii) $x(t)=e^{-t}u(t)$

$$R(\tau)=\int_{-\alpha}^{\alpha} e^{-t}u(t)e^{-(t\tau)}u(t-\tau)dt$$

$$=\int_{0}^{\infty} e^{-t}.e^{-t}.e^{t\tau}u(t-\tau)dt$$

$$=e^{\tau}\int_{\tau}^{\alpha} e^{-2t}dt$$

$$=e^{\tau}\frac{e^{-2\tau}}{2}=\frac{1}{2}e^{-\tau}$$

Now, lets find the energy of this signals

For $n(t)=e^{-t}u(t)$

$$E=\int_{-\alpha}^{\alpha}|x(t)|^2\,dt$$

$$=\int_{-\alpha}^{\alpha}|e^{-t}u(t)|^2\,dt$$

$$=\int_{0}^{\infty}e^{-2t}dt$$

$$=\left.\frac{e^{-2t}}{-2}\right]_0^{\alpha}=\frac{1}{2}$$

Since energy is finite and non-zero, it is a energy signal.

Now, for $x(t)=\sin\omega_o t$

$$t=\int_{-\alpha}^{\alpha}|x(t)|^2\,dt$$

$$=\int_{-T/2}^{T/2}\sin^2\omega_o t\,dt$$

$$=\int_{-T/2}^{T/2}\left(1-\frac{\cos 2\omega_o t}{2}\right)dt$$

$$=\frac{1}{2}\int_{-T/2}^{T/2}dt-\frac{1}{2}\int_{-T/2}^{T/2}\cos 2\omega_o t$$

In integration of second term will be zero since cosine wave over "full cycles", wherein integrated is equal to zero.

$\therefore \qquad E = \frac{T}{2}$

Hence energy is finite and non-zero, hence it is a energy signal.

Q. 2. (b) State and prove Parseval's theorem for power signals.

Ans. The theorem defines the power of a signal is terms of its fourier series coefficients, i.e., in terms of amplitudes of the harmonic components present in the signal.

Let us consider a function $f(t)$, we know that

$$|f(t)|^2 = f(t) f^*(t)$$

Where $f^*(t)$ is a complex conjugate of the function $f(t)$. The power of the signal $f(t)$ over a cycle is given by

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2\, dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t) f^*(t) dt$$

Replacing $f(t)$ by its exponential Fourier series

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)\left[\sum_{n\to-\alpha}^{\alpha} F_n e^{jn\omega_0 t}\right] dt$$

Interchanging the order of integration and summation

$$P = \frac{1}{T}\sum_{n\to\alpha}^{\alpha} F_n \int_{-T/2}^{T/2} f^*(t) e^{jn\omega_0 t}\, dt$$

The integral in the above expression is equal to TF_n^*.

Hence, we may write $\qquad P = \sum_{n\to-\alpha}^{\infty} f_n F_n^*$

$$= \sum_{n\to-\alpha}^{\alpha} |F_n|^2$$

This is Parseval's power theorem.

Q. 3. (a) A random process provides measurements x between the values 0 and 1 with a probability density function (PDF) given as :

$$f_X(x) = 12x^3 - 21x^2 + 10x \text{ for } 0 \le x \le 1$$
$$= 0 \text{ otherwise}$$

Determine the following :

(i) $P\left[X \le \frac{1}{2}\right]$ **&** $P\left[X > \frac{1}{2}\right]$

(ii) Obtain number k such that $P[X \le k] = \frac{1}{2}$.

Ans. (i)

$$f(x) = 12x^3 - 21x^2 + 10x \text{ for } 0 \le x \le 1$$

$$= 0 \qquad \text{otherwise}$$

$$P\left[X \le \frac{1}{2}\right] = \int_{-\alpha}^{1/2} f(u)du$$

But for $x \le 0$, $f(u) = 0$

$\therefore$

$$P = \left[X \le \frac{1}{2}\right] = 0$$

$$P\left[X > \frac{1}{2}\right] = \int_{1/2}^{\alpha} f(u)du$$

$$= \int_{1/2}^{1} (12u^3 - 21u^2 + 10u)du$$

$$= \left[\frac{12u^4}{4} - \frac{21u^3}{3} + \frac{10u^2}{2}\right]_{1/2}^{1}$$

$$= \left[\frac{12}{4} - \frac{21}{3} + \frac{10}{2} - \frac{12 \times (1/2)^4}{4} - \frac{21(1/2)^3}{3} + \frac{10(1/2)^2}{2}\right]$$

$$= \left[\frac{12}{4} - \frac{21}{3} + \frac{10}{2} - \frac{12}{64} - \frac{21}{24} + \frac{10}{8}\right]$$

$$= \left[\frac{192 + 320 - 12 + 80}{64} - \frac{21}{24} - \frac{21}{3}\right]$$

$$= \left[\frac{580}{64} - \frac{(21 + 168)}{24}\right]$$

$$= \left[\frac{580}{64} - \frac{189}{24}\right]$$

$$= [9.06 - 7.875]$$

$$= 1.185$$

(ii) $P[X \le k] = \frac{1}{2}$

$$\int_{-\alpha}^{k} f(u).d(u) = \frac{1}{2}$$

$$\int_{0}^{k} (12u^3 - 21u^2 + 10u)du = \frac{1}{2}$$

$$\left[\frac{12u^4}{4} - \frac{21u^3}{3} + \frac{10u^2}{2}\right]_0^k = \frac{1}{2}$$

$$3k^4 - 7k^3 - 5k^2 = \frac{1}{2}$$

$$k^2(3k^2 - 7k - 5) = \frac{1}{2}$$

Q. 3. (b) Define cumulative function. Explain different properties of cumulative distribution function.

Ans. The cumulative distribution function for a discrete random variable is defined as

$$F(X) = P(X \le x) = \sum_{u \le x} f(u) \qquad -\alpha < x < \infty$$

If X can take on the values $x_1, x_2, x_3 \ldots\ldots\ldots x_n$, then the distribution function is given by

$$f(x) = \begin{cases} 0, & -\alpha \le x < x_1 \\ f(x_1), & x_1 \le x < x_2 \\ f(x_1) + f(x_2), & x_2 \le x < x_3 \\ f(x_1) + f(x_2) + \ldots\ldots f(x_n), & x_n \le x < \infty \end{cases}$$

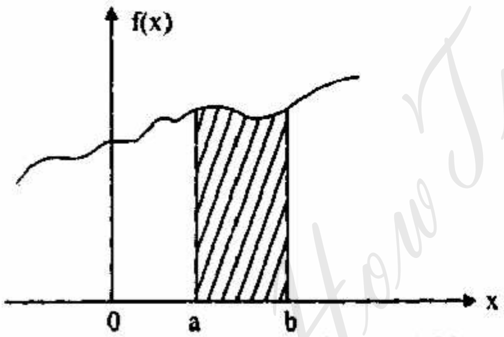
The cumulative distribution function for a continuous random variable is defined as :

$$F(x) = P(X \le x) = P(-\alpha < X \le x)$$

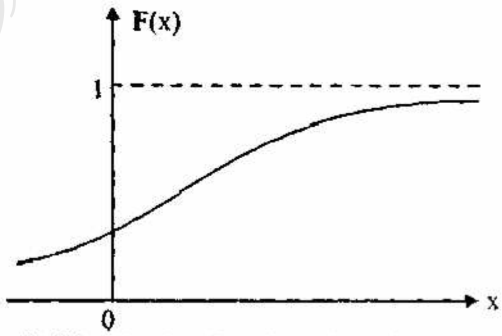
$$= \int_{-\alpha}^{x} f(u)du$$

At the points of continuity of $f(x)$, the sign $\le$ can be replaced by the sign $<$.

The plots of $f(x)$ & $F(x)$ are shown as :



(a) PDF of continuous random variable



(b) Distribution function of continuous random variable

Since $f(x) \ge 0$, the curve in figure (a) cannot fall below x-axis. The total area under the curve in figure (a) must be 1, and the shaded area in it gives the probability that x lies between a & b i.e., $P(a < x < b)$. The distribution function $F(x)$ is a monotonically increasing function that increases from 0 to 1 as shown in figure (b).

Q. 4. (a) Why is Gaussian distribution is most widely used? Explain.

Ans. Gaussian Distribution : This is the most important continuous probability distribution as most of the natural phenomenon are characterized by random variables with normal distribution. The importance of normal distribution is further enhanced because of central limit theorem. The density function for Gaussian distribution is given by

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2} \qquad -\alpha<x<\alpha$$

Where, μ & σ are mean and standard deviation, respectively. The properties of normal distribution are

$$\text{mean}=\mu \ \& \ \text{variance}=\sigma^2$$

The corresponding distribution function is

$$F(x)=P(X\leq x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\alpha}^{x}e^{-(v-\mu)^2/2\sigma^2}dv$$

$$=\frac{1}{2}+\frac{1}{\sigma\sqrt{2\pi}}\int_{\alpha}^{x}e^{-(v-\mu)^2/2\sigma^2}dv$$

Let Z be the standardized random variable corresponding to X. Thus, if $Z=\frac{x-\mu}{\sigma}$, then the mean of Z is zero and its variance is 1. Hence,

$$f(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

$f(z)$ is known as standard normal density function. The corresponding distribution function is :

$$F(z)=P(z\leq z)=\frac{1}{\sqrt{2\pi}}\int_{-\alpha}^{z}e^{-u^2/2}du$$

$$=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\int_{0}^{z}e^{-u^2/2}du$$

The integral is not easily evaluated. However, it is related to the error function, whose tabulated values are available in mathematical tables. The error function of z is defined as

$$erfz=\frac{2}{\sqrt{\pi}}\int_{0}^{z}e^{-u^2}du$$

The error function has values between 0 and 1

$$erf(0)=0, \quad \& \, erf(\alpha)=1$$

The complementary error function of z is defined as

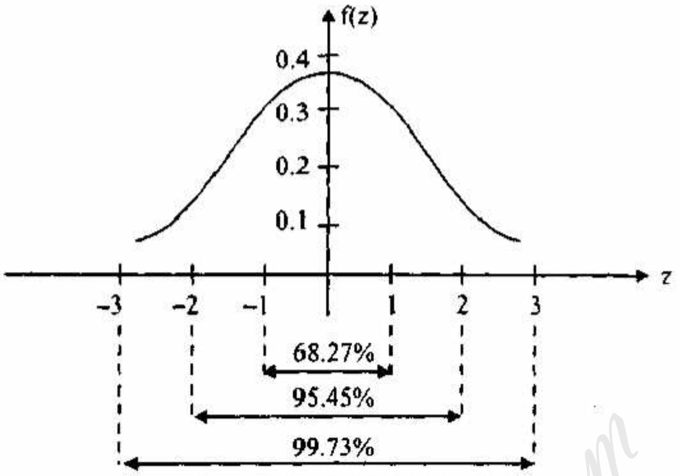
$$erfc(z)=1-erf(z)$$

$$=\frac{2}{\sqrt{\pi}}\int_{z}^{\alpha}e^{-u^2}du$$

The relationship between $F(z)$, $erf(z)$ & $erfc(z)$ is as follows :

$$F(z)=\frac{1}{2}\left[1+erf\left(\frac{z}{\sqrt{2}}\right)\right]=1-\frac{1}{2}erfc\left(\frac{z}{\sqrt{2}}\right)$$

The graph of $f(z)$ is known as standard normal curve is shown as :



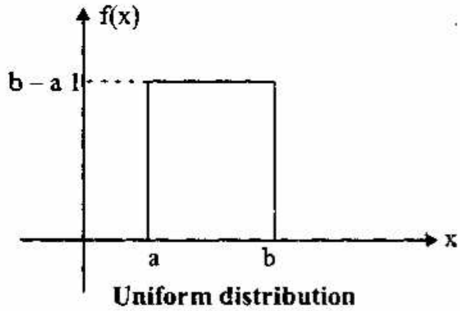
(Standard normal curve)

Q. 4. (b) Prove that the mean and variance of a random variable X having an uniform distribution in the interval $[a, b]$ are $m_x = \frac{a+b}{2}$ & $\sigma_x^2 = \frac{(a-b)^2}{12}$.

Ans. A random variable x is said to have a uniform distribution in the region $a \le x \le b$ if its density function is

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{ohterwise} \end{cases}$$

A uniform distribution is shown as



Uniform distribution

The properties are mean $= \frac{1}{2}(a+b)$

Variance $= \frac{1}{12}(b-a)^2$

Q. 5. (a) Define ergodic process. Explain the difference between ergodic process and stationary process.

Ans. To determine the statistics of the room temperature, say mean value, we may follow one of the following two procedures :

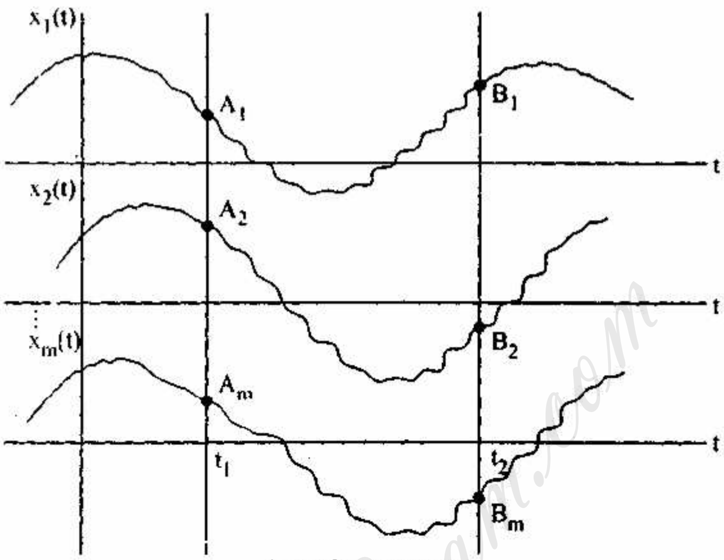
(i) We may fix t to some value, say t. The result is called, random variable $X(t_1, S) = X(t_1) = [A_1 A_2 \ldots\ldots A_m]$. The mean value of $X(t_1)$, $E[X(t_1)]$, can now be calculated. It is known as ensemble average. It may be noted that ensemble average is a function of time. There is an ensemble average corresponding to each time. Thus, at time t_2, we have

$$X(t_2, S) = X(t_2) = [B_1, B_2 \ldots\ldots B_m]$$

The ensemble average corresponding to time $t_2 = E[X(t_2)]$, can also be found out. Similarly, ensemble average corresponding to any time can be found out.

(ii) We may consider a sample function, say $x_1(t)$ over the entire time scale. Then the mean value of $x_1(t)$ is defined as

$$<x_1(t)> = \lim_{T \to \alpha} \frac{1}{2T} \int_{-T}^{T} x_1(t)dt$$



A random process

Similarly, we can find mean values of other sample functions. The expected value of all mean values is known as time average and is given as

$$<x(t)> = E[<x(t)>]$$

A random process for which mean values of all sample functions are same is known as a regular random process.

In this case, $<x_1(t)> = <x_2(t)> = ... = <x(t)> = <x(t)>$

For some process, ensembles average is independent to time, i.e.,

$$E[X(t_1)] = E[X(t_2)] = ... = E[X(t)]$$

Such processes are known as stationery processes in restricted sense. If all the statistical properties of a random process are independent to time, then it is known as stationary process in strict sense. When we say stationary process, then it is meant that the process is stationary in strict sense.

When an ensembled average is equal to the time average, then the process is known as ergodic process in restricted sense. When all the statistical ensemble properties are equal to statistical time properties, then the process is known as ergodic process in strict sense. When we say ergodic process, then it is meant that the process is ergodic in strict sense. It may be noted that ergodic process is a subset of a stationary process, i.e., if a process is ergodic, then it is also stationary, but the vice-versa is not necessarily true.

Q. 5. (b) Calculate the moments m_1, m_2 and S^2 for (i) Binomial distribution (ii) Gaussian distribution (iii) Rayleigh distribution.

Ans. The r^{th} moment of a random variable X about the origin is defined as

$$\mu'_r = E[(X)^r], \qquad r = 0, 1, 2$$

The r^{th} moment of a random variable X about the mean μ is defined as

$$\mu_r = E[(X-\mu)^r], \qquad r = 0, 1, 2$$

μ_r is also known as r^{th} central moment.

It can be seen that $\quad \mu'_0 = 1, \ \mu'_1 = \text{mean} = \mu$

$$\mu'_2 = \mu_2 + \mu^2$$

$$\mu_0 = 1, \ \mu_1 = 0, \ \mu_2 = \text{V or } (X)$$

In Discrete Case : i.e., for binomial distribution

$$\mu'_r = \sum_m x_m^r f(x_m)$$

&

$$\mu_r = \sum_m (x_m - \mu)^r f(x_m)$$

From binomial expansion, we have

$$(X-\mu)^r = \sum_{m=0}^{r} (-1)^m \binom{r}{m} X^{r-m} \mu^m$$

Hence

$$\mu_r = E[(X-\mu)^r]$$

$$= \sum_{m=0}^{r} (-1)^m \binom{r}{m} \mu^m E[X^{r-m}]$$

$$= \sum_{m=0}^{r} (-1)^m \binom{r}{m} \mu^m \mu'_{r-m} \qquad \text{.... (i)}$$

For Continuous Case : i.e., for Gaussian distribution

$$\mu'_x = \int_{-\alpha}^{\alpha} x^r f(x)dx$$

&

$$\mu_r = \int_{-\alpha}^{\alpha} (x-\mu)^2 f(x)dx$$

Moment Generating Function & Characteristic Function : The moment generating function of a random variable X is defined as :

$$M(t) = E[e^{tX}]$$

Now,

$$e^{tX} = 1 + tx + \frac{t^2X^2}{2!} + \ldots\ldots \frac{t^mX^m}{m!} + \ldots\ldots\ldots$$

$\therefore$

$$M(t) = E[e^{tX}] = E\left[1 + tX + \frac{t^2X^2}{2!} + \ldots\ldots\ldots + \frac{t^mX^m}{m!} + \ldots\ldots\right]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \ldots\ldots \frac{t^mE(X^m)}{m!} + \ldots\ldots$$

$$= 1 + t\mu'_1 + \frac{t^2}{2!}\mu'_2 + \ldots\ldots + \frac{t^m\mu'_m}{m!}$$

Differentiating about equation w.r.t. 't' and then putting $t = 0$, we get

$$\left.\frac{dM(t)}{dt}\right|_{t=0} = \mu'_1$$

Similarly differentiating m times w.r.t. t and then putting $t = 0$, we get

$$\left.\frac{d^m M(t)}{dt^m}\right|_{t=0} = \mu'_m$$

Moments about origin can be found out from the moment generating function by using the above equation. The movements about mean can then be found by using equation (i).

The characteristic function of a random variable X is defined as

$$\phi(\omega) = M(i\omega) = E[e^{i\omega x}], \quad i = \sqrt{-1}$$

By expanding this equation, as in the case of a moment generating function, it can be shown that :

$$\left.\frac{d^m\phi(\omega)}{d\omega^m}\right|_{\omega=0} = (i)^m \omega\mu'_m$$

Q. 6. (a) What is mutual information? How is it related to channel capacity?

Ans. Mutual Information : The state of knowledge at the receiver about the transmitted symbol x_j is the probability that x_j would be selected for transmission. This is a priori probability $P(x_j)$. After the reception and selection of the symbol y_k, the state of knowledge concerning x_j is the conditional probability $P(x_j/y_k)$ which is also known as a posteriori probability. Thus, before y_k is received, the uncertainity is $-\log P(x_j)$.

After y_K is received, the uncertainty becomes

$$-\log P(x_j/y_K)$$

The information gained about x_j by the reception of y_k is the net reduction in its uncertainty, and is known as mutual information $I(x_j, y_k)$. Thus,

$$I(x_j, y_k) = \text{Initial uncertainty} - \text{Final uncertainty}$$

$$= -\log P(x_j) - [\log P(x_j/y_k)] - \frac{\log P(x_j/y_k)}{P(x_j)}$$

$$= \frac{\log P(x_j, y_k)}{P(x_j)P(y_k)} = \frac{\log P(y_k/x_j)}{P(y_k)} = I(y_k, x_j)$$

Thus, we see that mutual information is symmetrical in x_j & y_k, i.e.,

$$I(x_j;\ y_k) = I(y_k;\ x_j)$$

Thus, $$I(x_j;\ x_j) = \frac{\log P(x_j/x_i)}{P(x_j)} = \log\frac{1}{P(x_j)}$$

$$= [I(x_j)]$$

Relation with Channel Capacity : The mutual information $I(X;\,Y)$ indicates a measure of the average information per symbol transmitted in the system. A suitable measure for efficiency of transmission of information may be introduced by comparing the actual rate and the upper bound of the rate of information transmission for a given channel. Shannon has introduced a significant concept of channel capacity defined as the maximum of mutual information. Thus, the channel capacity C is given by

$$C = \max I(X;\,Y)$$
$$= \max[H(X) - H(X/Y)]$$

Q. 6. (b) State and prove Shannon-Hartley theorem.

Ans. For a Gaussian channel, $P(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$

Hence, $$H(x) = -\int_{-\alpha}^{\alpha} P(x)\log P(x)dx$$

But $$-\log P(x) = \log\sqrt{2\pi\sigma^2} + \log e^{x^2/2\sigma^2}$$

Hence, $$H(x) = \int_{-\alpha}^{\alpha} P(x)\log\sqrt{2\pi\sigma^2}\,.dx + \int_{-\alpha}^{\alpha} P(x)\log e^{x^2/2\sigma^2}$$

This may be evaluated to yield

$$H(x) = \log\sqrt{2\pi e\sigma^2} \text{ bits/message}$$

Now, if the signal is bandwidth to ω Hz, then it may be uniquely specified by taking 200 samples per second. Hence, the rate of information transmission is

$$R(x) = 2\omega H(x)$$
$$= 2\omega\log\left(\sqrt{2\pi e\sigma^2}\right)$$
$$= \omega\log\left(\sqrt{2\pi e\sigma^2}\right)^2$$
$$R(x) = \omega\log(2\pi e\sigma^2)$$

If $P(x)$ is a bandlimited Gaussian noise with an average noise power N, then

$$R(n) = R(x) = \omega\log(2\pi eN) \qquad (\because \sigma^2 = N) \quad \text{.... (i)}$$

The transmitted signal with average power S and the noise on the channel be while Gaussian noise with an average power N with the bandwidth ω of the channel. The received signal will now have average power $(S+N)$. $R(y)$ is max when $y(t)$ is also a Gaussian random process. Thus, the entropy from equation (i) on per second basis is given as :

$$R(y) = \omega\log[2\pi e(S+N)] \text{ bits/sec.}$$

While the entropy of the noise is given by

$$R(n) = \omega\log(2\pi eN] \text{ bits/sec.}$$

The channel capacity may be obtained as

$$C = \max[R(y) - R(n)]$$
$$= \omega \log[2\pi e(S+N) - \omega \log(2\pi e N)]$$
$$= \omega \log\left[\frac{S+N}{N}\right]$$
$$C = \omega \log\left[1+\frac{S}{N}\right] \text{ bits/sec.}$$

This equation is famous Shannon-Hartley Theorem.

Q. 6. (c) A discrete memory less source produces five symbols with probabilities $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ & $\frac{1}{16}$ respectively. Determine the source entropy and information rate if the symbol rate is 10 Kbps.

Ans. The entropy H is

$$H = \sum_{k=1}^{5} P_k \log \frac{1}{P_k}$$
$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{16}\log 16 + \frac{1}{16}\log 16$$
$$= 0.1505 + 0.1505 + 0.1128 + 0.1505$$
$$= 0.5643 \text{ bits/sec.}$$

Now

$$r = 10 \text{ Kbps}$$
$$= 10 \times 10^3 \text{ bits/sec.}$$
$$= 10^4 \text{ bits/sec.}$$

$\therefore$ Rate of information R is

$$R = rH$$
$$= 10^4 \times 0.5643$$
$$= 5.643 \times 10^3 \text{ bits/sec.}$$

Q. 7. (a) Prove the following identities :

(i) $H(X,Y) = H(Y/X) + H(X)$

(ii) $H(X, Y) = H(X) + H(Y)$ if X and Y are statistically independent.

Ans. The relationship between the different entropies can be established as follows :

$$H(X,Y) = -\sum_{j=1}^{m}\sum_{k=1}^{n} P(x_j y_k)\log P(x_j, y_k)$$
$$= -\sum_{j=1}^{m}\sum_{k=1}^{n} P(x_j, y_k)\log[P(x_j/y_k)P(y_k)]$$

$$= -\sum_{j=1}^{m}\sum_{k=1}^{n} P(x_j, y_k)[\log P(x_j/y_k) + P(y_k)]$$

$$= -\sum_{j=1}^{m}\sum_{k=1}^{n} [P(x_j, y_k)\log P(x_j/y_k) + P(x_j, y_k)\log P(y_k)]$$

$$= H(X/Y) - \sum_{j=1}^{m}\sum_{k=1}^{n} P(x_j, y_k)\log P(y_k)$$

$$= H(X/Y) - \sum_{k=1}^{m}\left[\sum_{j=1}^{n} P(x_j, y_k)\right]\log P(y_k)$$

$$= H(X/Y) - \sum_{k=1}^{n} P(y_k)\log P(y_k)$$

$$= H(X/Y) + H(Y)$$

Similarly, it can be shown that

$$H(XY) = H(Y/X) + H(X)$$

For example, an experiment is performed 50 times, and the outcomes appears as given by the sequence below :

BAABBBBABAAABBAAAABABBABAABABAAABBAABABBBABABBAA

It is seen that $n = 50,\ n_A = 26,\ n_B = 24,\ n_{AB} = 14,\ n_{BA} = 15$

The different probabilities are as follows :

$$P(A) = \frac{n_A}{n} = \frac{26}{50} \qquad P_{(B)} = \frac{n_B}{n} = \frac{24}{50}$$

$$P(AB) = \frac{n_{AB}}{n} = \frac{14}{50} \qquad P(BA) = \frac{n_{BA}}{n} = \frac{15}{50}$$

$$P(B/A) = \frac{n_{AB}}{n_A} = \frac{14}{26} \qquad P(A/B) = \frac{n_{BA}}{n_B} = \frac{15}{24}$$

Now $P(AB) = P(A)P(B/A)$ gives

$$\frac{14}{50} = \frac{26}{50}\times\frac{14}{26},$$ Hence verified.

& $P(BA) = P(B)P(A/B)$ gives

$$\frac{15}{50} = \frac{24}{50}\times\frac{15}{24},$$ hence verified.

Now, let us consider a situation where the probability of the event B occuring is independent of the event A. Such a situation would be true n the two-card problem is the first card were immediately replaced after having been drawn.

In this case then $P(B/A) = P(B)$

Implying that the probability of event B is independent of event A

$\therefore$ $$P(AB) = P(A)P(B)$$

Similarly when probability of event A occuring is independent of event B, then

$$P(A/B) = P(A)$$

Implying that the probability of event A is independent of event B

$$P(BA) = P(B)P(A)$$

$$P(AB) = P(BA) = P(A)P(B)$$

The two events A & B are said to be statistically independent events if their probabilities satisfy the above equation.

Q. 7. (b) State and explain central limit theorem.

Ans. Central Limit Theorem : We know that both Binomial & Poisson distribution approaches normal distribution as the limiting case. This suggests that the sum of independent random variables will also approach normal distribution. The central limit theorem gives a statement in this direction which states that .

The probability density of a sum of N independent random variables will also approach normal distribution. The central limit theorem, the mean & variance of the normal density are the sums of mean and variance of N independent random variables. For example,

The electrical noise in communication system is due to a large number of randomly moving charged particles.

Hence according to central limit theorem, the instantaneous value of noise will have a normal distribution. As binomial distribution has relationship with both Poisson and normal distribution, one would expect that there should be some relationship between Poisson and normal distribution. In fact it is found to be so. It has been seen that the Poisson distribution approaches normal distribution as $\lambda \to \alpha$.

If n is large and if neither p not q is too close to zero, then the binomial distribution can be closely approximated by normal distribution with standardized random variable is given by

$$Z = \frac{X - np}{\sqrt{npq}}$$

In practice $np \geq 5$ and $nq \geq 5$ gives satisfactory performance.

Q. 8. Write short notes on any two :

(i) Cross Correlation function

(ii) Optimum filter

(iii) Error functions

Ans. (i) Cross Correlation Function : The correlation, or more precisely cross-correlations between two waveforms is the measure of similarity between one waveform and time-delayed version of the other waveform. This expresses how much one waveform is related to the time delayed version of the other waveform when scanned over time axis.

The expression for cross-correlation is very close to convolution. Consider two general complex functions $f_1(t)$ & $f_2(t)$ which may or may not be periodic and not restricted, to finite intervals. The cross-correlation or simply correlation $R_{12}(\tau)$ between two functions is defined as follows :

$$R_{12}(\tau) = \lim_{T \to \alpha} \int_{-T/2}^{T/2} f_1(t) f_2^*(t+\tau) dt$$

This represent the shift of function $f_2(t)$ by an amount $-\tau$ (i.e., towards left). A similar effect can be obtained by shifting $f_1(t)$ by an amount $+\tau$ (i.e., towards right). Hence, correlation may also be defined in an equivalent way, as

$$R_{12}(\tau) = \lim_{T \to \alpha} \int_{-T/2}^{T/2} f_1(t-\tau) f_2^*(t) dt$$

Let us define the correlation for two cases (i) energy (non-periodic) functions & (ii) power (periodic) functions. In the definition of correlation, limits of integration may be taken as infinite for energy signals.

$$R_{12}(\tau) = \int_{-\alpha}^{\alpha} f_1(t) f_2^*(t+\tau) dt$$

$$= \int_{-\alpha}^{\alpha} f_1(t-\tau) f_2^*(t) dt$$

For power signals of period T_0, may not converge. Hence, average correlation over a period T_0 is defined as :

$$R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t) f_2^*(t+\tau) dt$$

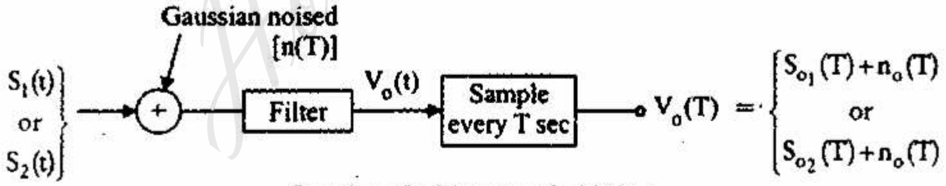
$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_1(t-\tau) f_2^*(t) dt$$

The correlation definition represents the overlapping area between the two functions.

(ii) Optimum Filter : In the receiver side, the signal is passed through a filter. It is important to known whether the filter is an optimum filter which gives the minimum probability of error.

If the input is $S_1(t)$, the output

$$V_o(T) = S_{o1}(T) + n_o(T)$$



Receive of a binary coded PCI 1

The decision boundary is, therefore $\dfrac{S_{o1}(T) + S_{o2}(T)}{2}$

An error will occur if at the sampling instant, the noise $n_o(T)$ is positive and larger than

$$\frac{1}{2}[S_{o1}(T)+S_{o2}(T)]-S_{o2}(T) \text{ or } \frac{S_{o1}(T)-S_{o2}(T)}{2}$$

Thus, an error will occur if

$$n_o(T) \geq \frac{S_{o1}(T)-S_{o2}(T)}{2}$$

Hence, the probability of error is

$$P_e = \int_{\frac{S_{o1}(T)-S_{o2}(T)}{2}}^{\alpha} \frac{e^{-n_0^2(T)/2\sigma_0^2}}{\sqrt{2\pi\sigma_0^2}} dn_o(T)$$

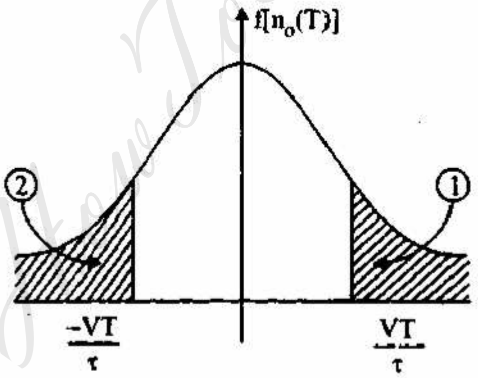
$$= \frac{1}{2} \times \frac{2}{\pi} \int_{\frac{S_{o1}(T)-S_{o2}(T)}{2}}^{\alpha} e^{-x^2} dx \quad \text{assuming } x = \frac{n_o(T)}{\sqrt{2\sigma_o}}$$

$$= \frac{1}{2} erfc\left[\frac{S_{o1}(T)-S_{o2}(T)}{2\sqrt{2}\sigma_o}\right]$$

P_e decreases as $\left[\frac{S_{o1}(T)-S_{o2}(T)}{2\sqrt{2}\sigma_o}\right]$ increases. The optimum filter is the filter which maximizes the ratio

$$Y = \frac{S_{o1}(T)-S_{o2}(T)}{\sigma_o}$$

(iii) Error Function : The probability density of the noise sample $n_o(T)$ is Gaussian as shown below. Hence



Probability density of noise sample n_o(T)

$$f[n_o(T)] = \frac{e^{-n_0^2(T)/2\sigma_0^2}}{\sqrt{2\pi\sigma_o^2}}$$

Since at the end of bit interval, $S_o T = \frac{-VT}{\tau}$, an error will occur only when $n_o(T) \geq \frac{VT}{\tau}$

$$P_e = \int_{VT/\tau}^{\alpha} f[n_o(T)]\, dn_o(T)$$

$$= \int_{VT/\tau}^{\infty} \frac{e^{-n_o^2(T)/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}}\, dn_o(T)$$

Substituting $x = n_o(T)/\sqrt{2}\,\sigma_o$, we get

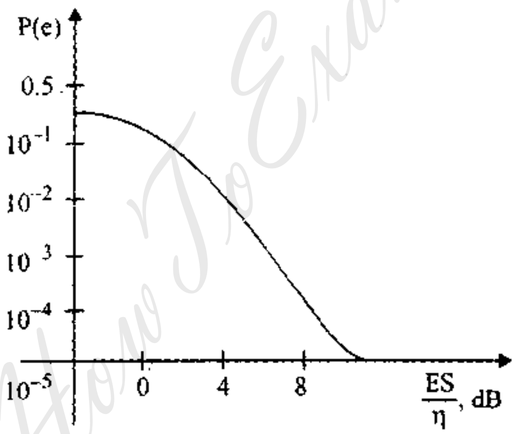
$$P_e = \frac{1}{2} \times \frac{2}{\sqrt{\pi}} \int_{x=V\sqrt{T/\eta}}^{\alpha} e^{-x^2} dx = \frac{1}{2} erfc(V\sqrt{T/\eta})$$

$$= \frac{1}{2} erfc\left(\frac{V^2 T}{\eta}\right)^{1/2} = \frac{1}{2} erfc\left(\frac{Es}{\eta}\right)^{1/2}$$

Also, at the end of the bit interval,

$$S_o(T) = \frac{VT}{\tau} \text{(for a1)}$$

Hence an error will occur if $n_o(T)$ is negative and is of a greater magnitude than $\frac{VT}{\tau}$. This error of probability is given by shaded area 2 in the above figure. The graph of $P(e) V/s \frac{Es}{\eta}$ in dB is shown as :



Note than P_e decreases rapidly as $\frac{ES}{\eta}$ increases.

The maximum value of P_e is 1/2. Thus, even if the signal is lost in the noise, the receiver cannot be wrong more than half the time on the average.