13. (a) Prove that a function ϕ is a solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$ on an interval of I if and only if it is a solution of the integral equation $y = y_0 + \int_{x_0}^{x} f(t, y)dt$ on I.

(b) Show that $f(x, y) = xy$ satisfies a Lipschitz condition on any strip $a \le x \le b, -\alpha < y < \alpha$ but not on the entire plane.

14. Describe Charpit's method for solving a first order partial differential equation. Use this method to solve $p^2x + q^2y = z$.

6559/KA4 **OCTOBER 2007**

Paper IV — NUMERICAL METHODS AND DIFFERENTIAL EQUATIONS

(For those who joined in July 2003 and after)

Time : Three hours Maximum : 100 marks

SECTION A — (4 × 10 = 40 marks)

Answer any FOUR questions.

Each question carries 10 marks.

1. (a) If $\phi(x)$ is a continuous function in some interval $[a, b]$ that contains the root and $|\phi'(x)| \le c < 1$ in this interval, then prove that for any choice of $x_0 \in [a, b]$, the sequence $\{x_K\}$ determined from $x_{K+1} = \phi(x_K), K = 0, 1, 2, \cdots$ converges to the root ξ of $x = \phi(x)$.

(b) Explain the Netwon-Raphson method.

2. (a) Explain Gauss elimination method.

(b) State and prove Brauer theorem.

3. (a) Derive the Hermite interpolating polynomial.

(b) Write down the properties of the Chebyshev polynomial $T_n(x)$.

4. (a) Define Lobotto Integration method and Radu Integration method.

(b) Derive the composite Simpson's rule formula.

5. (a) (i) State the Existence and Uniqueness theorem.

(ii) Define mesh points and mesh spacing.

(b) Explain Predictor-Corrector method.

6. If $\phi_1, \cdots, \phi_n$ be n solutions of

$$L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y = 0$$

on an interval I and if x_0 be any point in I, then prove that

$$W(\phi_1, \cdots, \phi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\phi_1, \cdots, \phi_n)(x_0).$$

7. Verify whether $y' = \frac{3x^2 - 2xy}{x^2 - 2y}$ is an exact equation and then solve it.

8. Find the general integral of the linear partial differential equation $z(xp - yq) = y^2 - x^2$

SECTION B — (3 × 20 = 60 marks)

Answer any THREE questions.

Each question carries 20 marks.

9. (a) Find the root of the equation $\cos x - x e^x = 0$ using the Secant and Regula-Falsi method.

(b) Find the largest eigen-value in modulus and the corresponding eigenvector of the matrix.

$$A = \begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$$

using the power method.

10. Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ using (a) composite trapezoidal rule (b) composite Simpson's rule, with 2, 4 and 8 equal subintervals.

11. Solve the initial value problem $u' = -2tu^2, u(0) = 1$ with $h = 0.2$ on the interval [0, 1] using the second order Runge Kutta method.

12. Solve the Bessel equation of order α, $x^2 y'' + xy' + (x^2 - \alpha^2) = 0, \operatorname{Re}\alpha \geq 0, \alpha \neq 0$.